Rotational Dynamics Chapter 9

9-1 Torque

In this chapter we follow the same progression we did for linear motion, that is, we're going to move from rotational *kinematics* to rotational *dynamics*. In making the transition to rotational kinematics, we replaced our linear kinematical variables with rotational variables. Likewise, we're going to make the same transition as we define new dynamical variables for rotation–force → *torque* and mass → *moment of inertia*, etc.

In this chapter we will consider only cases in which the rotational axis is in a fixed direction.

Torque as a scalar

Before we define the torque, we must do the following:

1. Identify a fixed axis of rotation.
2. Find the *moment arm*–the shortest distance between the axis of rotation and the point where the force is applied.
3. Calculate the amount of force perpendicular to the *moment arm*, sometimes called F *perp* or written as $F_{\perp}$.

The torque τ is defined as:

$$\tau = rF\sin\theta = rF_{\perp} \tag{1}$$

where r is the moment arm (measured in meters) and $F\sin\theta$ is the component of force perpendicular to the moment arm (measured in newtons). The units of torque τ are newton·meters (N·m)

Torque as a Vector

Looking at Eq. 1, we see that torque is the product of two quantities times $\sin\theta$. This suggests that we might be able to write the torque as a vector quantity $\vec{\tau}$.

$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{2}$$

where $\vec{\mathbf{r}}$ is a vector pointing from the *axis of rotation* to the point where the force $\vec{\mathbf{F}}$ is applied. More explictly the torque can be written in terms of x, y, and z components:

$$\vec{\tau} \;=\; (yF_z - zF_y)\hat{\imath} + (zFx - xF_z)\hat{\jmath} + (xF_y - yF_x)\hat{k}$$

Exercise 5: Two vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ lie in the xy plane. Their magnitudes are $r = 4.5$ units and $s = 7.3$ units. Their directions are, respectively, 320^o and 85^o measured counterclockwise from the positive x axis. Find the magnitude and the direction of $\vec{\mathbf{r}} \times \vec{\mathbf{s}}$.

9-2 Rotational Inertia and Newton's Second Law

If we consider the *axis of rotation* along the z axis, we observe that any component of force along the z axis does not contribute to the torque. The rotational inertia of a single particle is the *tangential* force $F\sin\theta$ which is equal to ma_T, where a_t is the tangential acceleration.

$$F\sin\theta = ma_T = mr\alpha$$

where α is the angular acceleration measured about the z-axis. Substituting this into Eq. 1 we find that the torque is:

$$\tau \;=\; rF\sin\theta \;=\; mr^2\alpha \;=\; I\alpha \qquad (3)$$

where I is called the *moment of inertia* and has units of kg·m^2. The moment of inertia of a single particle located a distance r from the axis of rotation is:

$$I \;=\; mr^2 \qquad (4)$$

Exercise 11: A small lead sphere of mass 25 g is attached to the origin by a thin rod of length 74 cm and negligible mass. The rod pivots about the z axis in the xy plane. A constant force of 22 N in the y direction acts on the sphere. *(a)* considering the sphere to be a particle, what is the rotational inertia about the origin? *(b)* If the rod makes an angle of 40^o with the positive x axis, find the angular acceleration of the rod.

Newton's Second Law for Rotation

Can we write a simple equation describing the motion of an extended rigid body (made up of many molecules) due to an applied force F? Fortunately, the answer is, "Yes." Your book works out the result for a force applied to a two-mass systems where the masses are held together by a massless rod and joined to the axis of rotation by massless rods. The result is simply $\sum \tau_z = (m_1 r_1^2 + m_2 r_2^2)\alpha_z$. This can be extended to an N-particle *rigid*body by writing:

$$\sum \tau_z \ = \ I\alpha_z \tag{5}$$

where

$$I = m_1\, r_1^2 + m_2\, r_2^2 + \cdots + m_N\, r_N^2$$

The Parallel-Axis Theorem

The *parallel-axis theorem* states:

> *The rotational inertia of any body about an arbitrary axis equals the rotational inertia about a parallel axis through the center of mass plus the total mass times the squared distance between the two axes.*

Mathematically, the parallel-axis theorem has the following form:

$$I = I_{\rm cm} + Mh^2 \tag{6}$$

where I is the *moment of inertia* about an arbitrary axis, $I_{\rm cm}$ is the moment of inertia for an axis pasing through the center-of-mass of the system, and h is the distance between the two parallel axes.

Example: Calculate the moment of inertia for a two-mass system (of equal masses) separated by a distance d *(a)* about an axis through its center of mass where the axis is perpendicular to the massless bar joining the two masses, and *(b)* about an axis through one of the masses (using the *parallel axis theorem.*

9-3 Rotational Inertia of Solid Bodies

The moment of inertia can be calculated for an extended object (i.e., non-point-like objects). See Fig. 9-15 in this chapter for a list of moments of inertia.

Example: Calculate the moment of inertia for a sphere of mass M and radius R about an axis tangential to its surface.

Exercise 17: Fig. 9-43 shows a uniform block of mass, M and edge lengths a, b, and c. Calculate its rotational inertia about an axis through one corner and perpendicular to the large face of the block. (Hint: See Fig. 9-15.)

Exercise 20: *(a)* Show that a solid cylinder of mass M and radius R is equivalent to a thin hoop of mass M and radius $r/\sqrt{2}$, for rotation about a central axis. *(b)* The radial distance from a given axis at which the mass of a body could be concentrated without altering the rotational inertia of the body about that axis is called the *radius of gyration*. Let k represent the radius of gyration and show that

$$k = \sqrt{\frac{I}{M}}$$

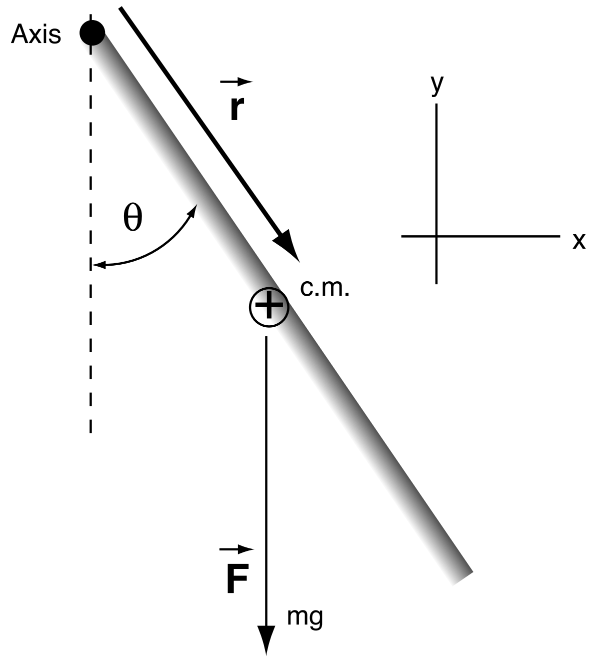
This gives the radius of the "equivalent hoop" in the general case.

9-4 Torque Due to Gravity

Imagine that we have a beam whose axis of rotation is at one end of the beam. We can calculate the torque "just as if" all the mass were concentrated at its *center of mass*. This is much more convenient than calculating the torque due to gravity acting on every molecule of the beam. The torque on the beam is:

$$\vec{\tau} = \vec{\mathbf{r}}_{\rm cm} \times \vec{\mathbf{F}}$$

where $\vec{\mathbf{F}}$ is the weight vector having magnitude mg and pointing toward the center of the earth..



Use the right-hand-rule to determine which way the torque is acting on the beam. If the angular acceleration $\vec{\alpha}$ is proportional to the external torque $\vec{\tau}$, what can you say about the "sense" of rotation for the above bar?

Center-of-mass vs. Center-of-gravity

Describe the difference between the *center of mass* and the *center of gravity*. These concepts are one-in-the-same if the gravitational field is uniform over the extent of the object. However, they will be different if the gravitational field is not uniform over the extent of the object.

9-5 Equilibrium Applications of Newton's Laws for Rotation

We now have two conditions of equilibrium:

$$\sum \vec{\mathbf{F}}_{\text{ext}} = 0 \qquad \text{for translational motion}$$

and

$$\sum \vec{\tau}_{\text{ext}} = 0 \qquad \text{for rotational motion.}$$

Each of these vector equations represents three scalar equation–one in each of the x, y, and z directions.

Equilibrium Analysis Procedures

1. Draw a boundary around the system.
2. Draw a free-body diagram showing all external forces that act on the system and their points of application.
3. Set up a coordinate system and choose the direction of the axes.
4. Set a coordinate system and axis of rotation for resolving the torques into their components.

Exercise 25: In Sample Problem 9-7 the coefficient of static frction μ_{s} between the ladder and the ground is 0.54. How far up the ladder can the firefighter go before the ladder starts to slip?

Sample Problem 9-7: A ladder whose length L is 12 m and whose mass m is 45 kg rests against a wall. Its upper end is a distance h of 9.3 m above the ground, as in Fig. 9-23a. The center of mass of the ladder is one-third of the way up the ladder. A firefighter whose mass M is 72 kg climbs halfway up the ladder. Assume that the wall, but not the ground, is frictionless. What forces are exerted on the ladder by the wall and by the ground?

Exercise 29: What minimum force F applied horizontally at the axle of the wheel in Fig. 9-49 is necessary to raise the wheel over an obstacle of height h? Take r as the radius of the wheel and W as its weight.

9-6 Nonequilibrium Applications of Newton's Laws for Rotation

We previously wrote Newton's 1st Law for rotational equilibrium ($\sum \vec{\tau}_{\text{ext}} = \vec{0}$). Now we are ready to explore solutions to systems that are not in rotational equilibrium, that is, they obey Newton's 2nd law for rotational motion:

$$\sum \vec{\tau}_{\text{ext}} = I\vec{\alpha} \tag{7}$$

Exercise 37: A pulley having a rotational inertia of 1.14×10^{-3} kg·m^2 and a radius of 9.88 cm is acted on by a force, applied tangentially at its rim, that varies in time as $F = At + Bt^2$, where $A = 0.496$N/s and $B = 0.305$ N/s^2. If the pulley was initially at rest, find its angular speed after 3.60 s.

9-7 Combined Rotational and Translational Motion

The most common example of combined rotational and translational motion is that of a wheel rolling on the ground or on an incline. Let's look at Sample Problem 9-11 in more detail.

Sample Problem 9-11

> A solid cylinder of mass M and radius R starts from rest and rolls without slipping down an inclined plane of length L and height h (Fig. 9-32). Find the speed of its center of mass when the cylinder reaches the bottom.

Exercise 41: An automobile traveling 78.3 km/h has tires of 77.0-cm diameter. *(a)* What is the angular speed of the tires about the axle? *(b)* If the car is brought to a stop uniformly in 28.6 turns of the tires (no skidding), what is the angular acceleration of the wheels? *(c)* How far does the car advance during this braking period?

Problem 3: A uniform sphere of weight W and radius r is being held by a rope attached to a frictionless wall a distance L above the center of the sphere, as in Fig. 9-58. Find *(a)* the tension in the rope and *(b)* the force exerted on the sphere by the wall.