Motion in Two and Three Dimensions   Chapter 4

4-1 Motion in Three Dimensions with Constant Acceleration

If the accelerations in all three dimensions are constant, (i.e., $a_x = \text{constant}_1$, $a_y = \text{constant}_2$, and $a_z = \text{constant}_3$), then the velocity and position vectors can be written in vector form:

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_o + \vec{\mathbf{a}}t \tag{1}$$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_o + \vec{\mathbf{v}}t + \frac{1}{2}\vec{\mathbf{a}}\,t^2 \tag{2}$$

Recall that the vector equations really represent 3 scalar equations. So, for Eq. 1, where $\vec{\mathbf{v}} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{\mathbf{v}}_o = v_{ox}\hat{\imath} + v_{oy}\hat{\jmath} + v_{oz}\hat{k}$, we have:

$$v_x = v_{ox} + a_x t \tag{3}$$
$$v_y = v_{oy} + a_y t \tag{4}$$
$$v_z = v_{oz} + a_z t \tag{5}$$

Likewise, for Eq. 2, there are also 3 scalar equations where $\vec{\mathbf{r}} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{\mathbf{r}}_o = x_o\hat{\imath} + y_o\hat{\jmath} + z_o\hat{k}$.

$$x = x_o + v_{ox}\,t + \frac{1}{2}a_x\,t^2 \tag{6}$$
$$y = y_o + v_{oy}\,t + \frac{1}{2}a_y\,t^2 \tag{7}$$
$$z = z_o + v_{oz}\,t + \frac{1}{2}a_z\,t^2 \tag{8}$$

Exercise 1: In a cathode-ray tube, a beam of electrons is projected horizontally with a speed of 9.6×10^8 cm/s into the region between a pair of horizontal plates 2.3 cm long. An electric field between the plates causes a constant downward acceleration of the electrons of magnitude 9.4×10^{16} cm/s^2. Find *(a)* the time required for the electrons to pass through the plates, *(b)* the vertical displacement of the beam in passing through the plates, and *(c)* the horizontal and vertical components of the velocity of the beam as it emerges from the plates.

4-2 Newton's Laws in Three-Dimensional Vector Form

As we mentioned in the previous chapter, the vector form of Newton's 2$^{\text{nd}}$ law:

$$\sum \vec{\mathbf{F}}_{ext} = m\vec{\mathbf{a}}$$

where $\vec{\mathbf{a}} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$, can be written as three scalar equations:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \text{and} \qquad \sum F_z = ma_z \ .$$

The same interptation can be applied to Newton's 3$^{\text{rd}}$ law, namely, $\vec{\mathbf{F}}_{\text{AB}} = -\vec{\mathbf{F}}_{\text{BA}}$ can be written as three scalar equations. Each component of the force in the x, y, and z direction must also *balance.*

$$(F_{\text{AB}})_x = -(F_{\text{BA}})_x \qquad (F_{\text{AB}})_y = -(F_{\text{BA}})_y \qquad (F_{\text{AB}})_z = -(F_{\text{BA}})_z$$

Exercise 7: A 5.1-kg block is pulled along a frictionless floor by a cord that exerts a force $P = 12$ N at an angle $\theta = 25^o$ above the horizontal, as shown in Fig. 4-29. *(a)* What is the accelertion of the block? *(b)* The force P is slowly increased. What is the value of P just before the block is lifted off the floor? *(c)* What is the acceleration of the block just before it is lifted off the floor?

Exercise 12: A jet fighter takes off at an angle of 27.0^o with the horizontal, accelerating at 2.62 m/s^2. The weight of the plane is 79,300 N. Find *(a)* the thrust T of the engine on the plane and *(b)* the lift force L exerted by the air perpendicular to the wings; see Fig. 4-34. Ignore air resistance.

4-3 Projectile Motion

A common example of two-dimensional motion is *projectile* motion. In this case, we will restrict the range of motion to the $x - y$ plane with the no acceleration along the x direction ($a_x = 0$) and constant acceleration along the y direction ($a_y = -g$). Using Eqs. 3 and 6, we can see that the motion along the x direction can be described by

$$v_x = v_{ox} \qquad \text{and} \qquad x = x_o + v_{ox}t$$

while the motion along the y direction can be described by Eqs. 4 and 7

$$v_y = v_{oy} + a_y t \qquad \text{and} \qquad y = y_o + v_{oy}t - \frac{1}{2}gt^2$$

the parametric equations $x(t)$ and $y(t)$ share one common kinematical variable (t), so a relation between y and x can be obtained. If we assume $x_o = 0$ and $y_o = 0$ at $t_o = 0$, then we find:

$$y = x \tan\phi_o - \frac{1}{2}\frac{g}{v_{ox}^2}x^2 \tag{9}$$

the equation of an "upside-down" parabola. Therefore, projectile motion (assuming no air-resistance) follows a parabolic trajectory in the x-y plane.

The projectile begins its motion with a velocity v_o at an angle θ_o, where $\tan\phi_o = v_{oy}/v_{ox}$. The projectile lands with a velocity $v = \sqrt{v_x^2 + v_y^2}$ at angle ϕ, where $\tan\phi = v_y/v_x$.

The *horizontal range* R of the projectile is defined as the distance along the horizontal where the projectile returns to the level from which it was launched. The range R can be found by setting $y = 0$ in Eq. 9 and solving for x and equating x to the range R.

$$R = \frac{v_o^2 \sin 2\phi_o}{g} \qquad \text{(on level ground only !!)} \tag{10}$$

Question: At what angle do you obtain the maximum range?

Question: If the range R is determined by Eq. 10, what other angle besides ϕ_o will result in the same range.

Exercise 14: Electrons, like all forms of matter, fall under the influence of gravity. If an electron is projected horizontally with a speed of 3.0×10^7 m/s (one-tenth the speed of light), how far will it fall in traversing 1.0 m of horizontal distance?

Exercise 23: A football player punts the football so that it will have a "hang time" (time of flight) of 4.50 s and land 50 yd (=45.7 m) away. If the ball leaves the player's foot 5.0 ft (=1.52 m) above the ground, what is its initial velocity (magnitude and direction)?

4-4 Motion in Drag Forces and the Motion of Projectiles

Suppose the *drag* force on a body falling vertically through the air (with the $+y$ axis pointing downward) is $D = -bv_y$, where b is a positive constant. Using Newton's 2^{nd} law in the y direction we obtain :

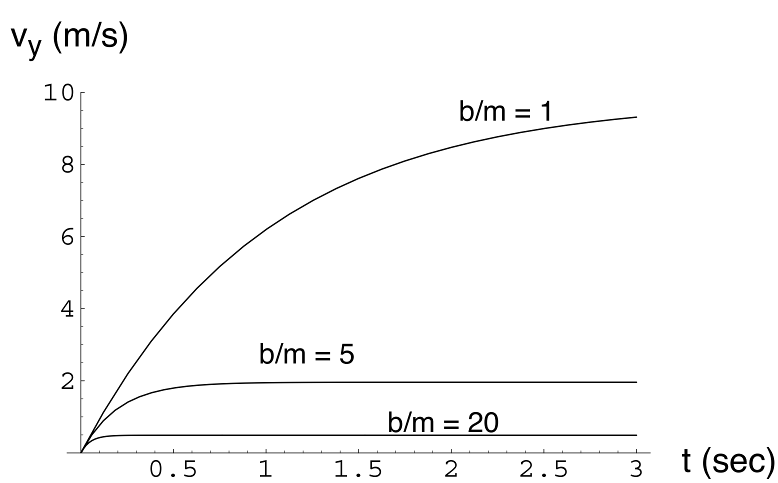
$$mg - bv_y = ma_y$$

or

$$a_y = g - \frac{bv_y}{m}$$

This leads to a simple first-order differential equation using $a_y = dv_y/dt$. To obtain the solution to this equation requires calculus, however, the result can be simply stated as:

$$v_y(t) \;=\; \frac{mg}{b}\left(1 - e^{-bt/m}\right)$$



Exercise 29: A small 150-g pebble is 3.4 km deep in the ocean and is falling with a constant terminal speed of 25 m/s. What force does the water exert on the falling pebble?

4-5 Uniform Circular Motion

Now we study the acceleration due to an object moving with uniform speed v in a cirucular path of radius r. The magnitude of the velocity vector v (*the speed*) is constant, however, the direction of the velocity vector $\vec{\mathbf{v}}$ is continuously changing. If we calculate the average acceleration vector $\vec{\mathbf{a}}_{av}$, the acceleration appears to point to the center of the circle with a magnitude of v^2/r.

$$a_{av} = \frac{v^2}{r} \qquad a_c = \frac{v^2}{r} \qquad \text{(the centripetal acceleration)} \tag{11}$$

Up until now, we have assumed the acceleration on the *right-hand-side* of Newton's 2^{nd} law was acceleration due to a constant acceleration vector $\vec{\mathbf{a}}$. Here is our first example of how a non-constant acceleration vector appears in Newton's 2^{nd} law.

$$\sum F_x = ma_x = m\frac{v^2}{r} \tag{12}$$

where mv^2/r is called the *centripetal force*.

Exercise 34: In Bohr's model of the hydrogen atom, an electron moves about the proton in a circular orbit of radius 5.29×10^{-11} m with a speed of 2.18×10^6 m/s. *(a)* What is the acceleration of the electron in this model of the hydrogen atom? *(b)* What is the magnitude and direction of the net force that acts on the electron?

4-6 Relative Motion

For now, we will limit our study of relative motion to positions, velocities and accelerations measured between two *inertial frames*. Recall that observers in two inertial frames will record the same acceleration of an object moving through space.

The transformation of velocities is an easy concept to understand and is best demonstrated by an example. Suppose the velocity of a jet *with respect to the air* is $\vec{\mathbf{v}}_{JA}$ and the velocity of the air *with respect to the ground* is $\vec{\mathbf{v}}_{AG}$, then the velocity of the jet with respect to the ground can be written as:

$$\vec{\mathbf{v}}_{JG} = \vec{\mathbf{v}}_{JA} + \vec{\mathbf{v}}_{AG} \tag{13}$$

This is called the *transformation of velocities*, or the *Galilean form of the law of transformation of velocities.*

Notice that if the *ground* and the *air* are two inertial frames, then the acceleration of the jet with repsect to the ground $\vec{\mathbf{a}}_{JG}$ and the acceleration of the jet with respect to the air $\vec{\mathbf{a}}_{JA}$ are the same

$$\vec{\mathbf{a}}_{JG} = \vec{\mathbf{a}}_{JA}$$

because $\vec{\mathbf{a}}_{AG} = 0$ (i.e., $\vec{\mathbf{v}}_{AG}$ is a constant vector).

Exercise 41: A transcontinental flight at 2700 mi is scheduled to take 50 min longer westward than eastward. The air speed of the jet is 600 mi/h. What assumptions about the jet-stream wind velocity, presumed to be east or west, are made in preparing the schedule?