

The Wave Function Chapter 1

1-1 The Schrodinger Equation

$$E_{tot} = KE + PE$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \quad (1)$$

We don't see any *quantum* effects until we make measurements at small distances (in most cases).

$$\hbar = 1.054572 \times 10^{-34} \text{ J} \cdot \text{s}$$

The wave function is complex in general.

1-2 The Statistical Interpretation

Max Born's statistical interpretation gives us a method for interpreting the wave function. While the wave function is the solution to the Schrodinger Equation, it is the "wave function squared" that is a *measurable* quantity.

$$\Psi^*(x, t) \Psi(x, t) = \text{probability density}$$

$$\int_a^b |\Psi(x, t)|^2 dx = \left\{ \begin{array}{l} \text{probability of finding the particle between } a \\ \text{and } b, \text{ at time } t. \end{array} \right\} \quad (2)$$

The *probability* of finding the particle at location x at time t is the area under the curve $\Psi^*(x, t) \Psi(x, t)$.

1-3.1 Probability for Discrete Variables

There are certain questions we can ask in quantum mechanics that give "reasonable" answers.

We can ask about **the mean value** or **expectation value**.

$$\langle j \rangle = \sum_{j=0}^{\infty} j P(j) \quad \text{where} \quad \sum_{j=0}^{\infty} P(j) = 1 \quad \text{where} \quad P(j) = \frac{N(j)}{N}$$

where

$$N = \sum_{j=0}^{\infty} P(j)$$

We can ask about **the spread of a distribution**. This is sometimes called the **standard deviation**.

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

where

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j)$$

1.3.2 Probability for Continuous Variables

The probability that x lies between a and b is:

$$P_{ab} = \int_a^b \rho(x) dx$$

where $\rho(x)$ is the **probability density**.

$$\rho(x) dx = \left. \begin{array}{l} \text{probability that an event } x \text{ (chosen at random)} \\ \text{lies between } x \text{ and } (x + dx). \end{array} \right\}$$

The rules we described for calculating statistical quantities for discrete variables “carry over” to calculating the same quantities for continuous variables.

$$\int_{-\infty}^{\infty} \rho(x) dx = 1$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \rho(x) dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x) dx$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x) \rho(x) dx$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Problem 1.3 Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where A , a , and λ are positive real constants. (Look up any integrals you need.)

- (a). Use Equation 1.16 to determine A .
- (b). Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ .
- (c). Sketch the graph of $\rho(x)$.

Problem 1.5 Consider the wave function

$$\Psi(x, t) = Ae^{-\lambda|x|} e^{-i\omega t}$$

where A , λ , and ω are positive real constants. (We'll see in Chapter 2 what potential (V) actually produces such a wave function.)

- (a) Normalize Ψ .
- (b) Determine the expectation values of x and x^2 .
- (c) Find the standard deviation of x . Sketch the graph of $|\Psi|^2$, as a function of x , and mark the points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$, to illustrate the sense in which σ represents the “spread” in x . What is the probability that the particle would be found outside this range?

1.4 Normalization

In order for the statistical interpretation of the $\Psi^*\Psi$ to make sense, we must be able to normalize the wavefunction “over all space”, that means from $-\infty$ to $+\infty$. The mathematical procedure for stating this is:

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1 \tag{3}$$

Most wavefunctions are written in the form of

$$\Psi(x, t) = Af(x, t)$$

where A is called the *amplitude* or *normalization* of the wavefunction. Solutions to the Schrodinger equation (i.e., $\Psi(x, t)$), are in general, not normalized. We have to “crank” them through Eq. 3 in order to determine their normalization. Once the correct value of A has been determined and $\Psi(x, t)$ satisfies Eq. 3, then $\Psi(x, t)^*\Psi(x, t)$ can be interpreted as a normalized probability density $\rho(x, t)$.

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1.5 Momentum

Classically, the momentum is defined as $p = mv$. However, it’s not clear what velocity *means* in quantum mechanics. We know how to calculate the *mean position* of an ensemble of identical experiments:

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 dx$$

We can also describe how the mean value of the position moves with time by calculating

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{m} \int \Psi^* \frac{\partial \Psi}{\partial x} dx$$

Finally, we can calculate the “net momentum” of an ensemble of identically prepared quantum systems:

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = -i\hbar \int \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) dx$$

This leads us to a more general (more suggestive) way of writing the *expectation values* of dynamical quantities:

$$\langle x \rangle = \int \Psi^* (x) \Psi dx \quad (4)$$

$$\langle p \rangle = \int \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi dx \quad (5)$$

So, in general, the expectation value of any dynamical quantity can be written as:

$$\langle Q(x, p) \rangle = \int \Psi^* Q \left(x, \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi dx \quad (6)$$

1.6 The Uncertainty Principle

Now that we’ve defined the spread of a probability density function using our definition of the *standard deviation* (σ), we can write the Heisenberg Uncertainty Principle in terms of the spread in position and momentum with the following equation:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \quad (7)$$

where $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$.

Problem 1.9: A particle of mass m is in the state

$$\Psi(x, t) = A e^{-a[(mx^2/\hbar)+it]}$$

where A and a are positive real constants.

- (a) find A .
- (b) For what potential energy function $V(x)$ does Ψ satisfy the Schrodinger equation?
- (c) Calculate the expectation values of x , x^2 , p , and p^2 .
- (d) Find σ_x and σ_p . Is their product consistent with the uncertainty principle?