Chapter 14
Periodic Motion

1 Describing Oscillation

First, we want to describe the kinematical and dynamical quantities associated with Simple Harmonic Motion (SHM), for example, x, v_x, a_x, and F_x.

x the **displacement** from equilibrium

F_x the restoring force $F_x = -kx$ "Hooke's Law"

A the **amplitude**, the maximum distance from equilibrium–measured in m

T the **period**, the time for one complete oscillation–measured in s

f the **frequency**, the number of oscillations (complete cycles) / sec – measured in *Hz*

and

ω the **angular frequency**, measured in *rad/s*

Example of Harmonic Motion

Force vs. Displacement

Ex. 3 The tip of a tuning fork goes through 440 complete vibrations in 0.500 s. Find the angular frequency and the period of the motion.

2 Simple Harmonic Motion

The restoring force exerted by an *ideal* spring is described by Hooke's Law–$F_x = -kx$. When the restoring force is direction proportional to the displacement from equilibrium, the oscillation is called **simple harmonic motion**.

From Newton's second law, we can write $\sum F_x = ma_x$.

$$-k\,x \;=\; m\frac{d^2x}{dt^2}$$

The solution to this differential equation is quite simple:

$$x(t) \;=\; A\cos(\omega t) \qquad \text{or more generally} \qquad x(t) \;=\; A\cos(\omega t + \phi)$$

where ϕ is called the phase angle. The value of ϕ depends on the position of the oscillator at time $t = 0$. Furthermore, the angular frequency is simply $\sqrt{k/m}$, and from this we can obtain f and T.

$$f = \frac{\omega}{2\pi} \qquad \text{and} \qquad T = \frac{1}{f}$$

2.1 Velocity and acceleration from $x(t)$

Once the equation of motion is known, it's simple to find the velocity and acceleration as a function of time (i.e., $v(t) = dx/dt$ and $a(t) = dv/dt$).

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}A\cos\omega t = -A\omega\sin\omega t$$

where $v_{\text{max}} = A\omega$, and

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(-A\omega\sin\omega t\right) = -A\omega^2\cos\omega t$$

where $a_{\text{max}} = A\omega^2$.

Ex. 9 When a body of unknown mass is attached to an ideal spring with force constant 120 N/m, it is found to vibrate with a frequency of 6.00 Hz. Find a) the period; b) the angular frequency; c) the mass of the body.

Ex. 15 The point of the needle of a sewing machine moves in SHM along the x-axis with a frequency of 2.5 Hz. At $t = 0$ its position and velocity components are +1.1 cm and -15 cm/s. a) Find the acceleration component of the needle at $t = 0$. b) Write equations giving the position, velocity, and acceleration components of the point as a function of time.

3 Energy in Simple Harmonic Motion

Energy principles (esp. the conservation of energy) apply to simple harmonic motion as well. For example, let's calculate the total energy of a simple harmonic oscillator.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Substituting $x = A\cos\omega t$ and $v = -A\omega\sin\omega t$ into the above equation, we find that:

$$E = \frac{1}{2}kA^2 \qquad \text{(the total energy of a SHO)}$$

which is independent of time. In other words, E, the energy is a constant of the motion.

Another reason for applying energy principles to a harmonic oscillator is to determine the velocity at any point in the motion. Let's say you know the *frequency* and *amplitude*, and you're given the value x, (the distance from the equilibrium position). How would you determine the velocity, v? From conservation of energy we have:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Solving this equation for v, we find:

$$v = \pm\omega\sqrt{A^2 - x^2} \qquad \text{(velocity at a point } x\text{)}$$

If we want to find the acceleration as a function of position (x), then we go back to Newton's 2nd Law ($-kx = ma$), and find that:

$$a = -\frac{k}{m}x = -\omega^2 x \qquad \text{(acceleration at a point } x\text{)}$$

Ex. 28 A harmonic oscillator has angular frequency ω and amplitude A. a) What are the magnitudes of the displacement and velocity when the elastic potential energy is equal to the kinetic energy? (Assume that $U = 0$ at equilibrium.) b) How often does this occur in each cycle? What is the time between occurrences? c) At an instant when the displacement is equal to $A/2$, what fraction of the total energy of the system is kinetic and what fraction is potential?

4 Applications of Simple Harmonic Motion

In this section we investigate the simple harmonic motion for a vertical oscillator (i.e., gravity is one of the external forces), and a torsion oscillator (where gravity has no effect).

4.1 Vertical SHM

When a mass is attached to the end of a vertical spring and gently released, it will stretch the spring a distance $\Delta\ell$. This location becomes the new equilibrium point about which simple harmonic motion occurs. At this new equilibrium point, we can write the following from Newton's 1st Law:

$$\sum F_x = 0 \qquad -mg + k\Delta\ell = 0 \qquad mg = k\Delta\ell$$

N.B. This same point is **not** the equilibrium point for the spring when calculating its potential energy. If you ever apply energy considerations to a vertical spring, be careful not to confuse these two equilibrium positions.

The equation of motion is the same as before. Let's assume that $+x$ is in the upward direction. Then,

$$x = A\cos(\omega t + \phi)$$

where $\omega = \sqrt{k/m}$ and ϕ is the phase angle.

Ex. 38 A proud deep-sea fisherman hangs a 65.0-kg fish from an ideal spring having negligible mass. The fish stretches the spring 0.180 m. (a) Find the force constant of the spring? The fish is now pulled down 5.00 cm and released. (b) What is the period of oscillation of the fish? c) What is the maximum speed it will reach?

4.2 Angular SHM

Suppose we have a torsion balance (a horizontal disk suspended by a thin wire) and we set it into oscillation. Instead of the spring constant k, we will define the *torsion* constant κ (kappa) and write Hooke's law as $\tau = -\kappa\theta$, where θ is the angular displacement from equilibrium. Applying Newton's 2^{nd} law to this problem we find:

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

We can write this last equation in the form of a SHO equation in the following way:

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

where $\omega^2 = \kappa/I$. The equation of motion for the torsion pendulum can be written as:

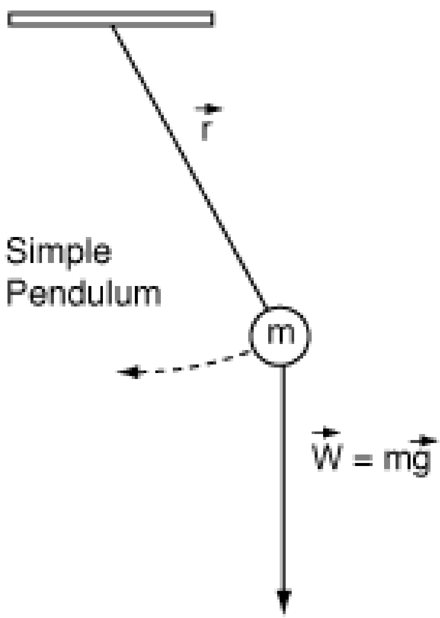
$$\theta(t) = \theta_o \cos(\omega t + \phi)$$

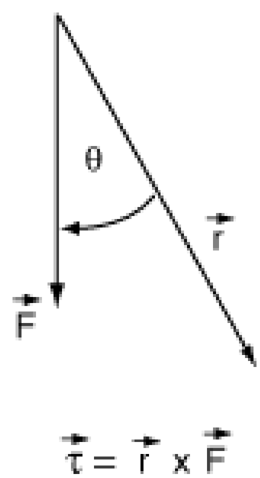
where θ_o is the angular amplitude, and $\omega = \sqrt{\kappa/I}$.

5 The Simple Pendulum

A common tool for keeping time is the simple, or physical pendulum. In this section we will investigate the mechanical motion of a simple pendulum where one end of a massless, rigid rod rotates about a pivot held at the top, and all the mass in the system is presumed to be in a sphere at the other end. The oscillations occur in a vertical plane about an equilibrium position defined by a plumb line.

The component of gravitational force is in the direction of the circular arc traced by the motion of the mass at the end of the rod, $mg\sin\theta$. Likewise, the acceleration it produces is d^2s/dt^2, where $s = L\theta$. Using Newton's 2^{nd} law, we can find the equation of motion for this system.





$$F_x = m\,a_x \qquad \Rightarrow \qquad -mg\sin\theta = m\frac{d^2s}{dt^2}$$

$$-\,mg\sin\theta = mL\frac{d^2\theta}{dt^2} \qquad \Rightarrow \qquad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

This equation almost looks like our famous SHO equation $\left(d^2x/dt^2 + \omega^2 x = 0\right)$. In it's present form it can only be solved by using elliptic functions. However, if we make the small-angle approximation $\sin\theta \approx \theta$, then we can write Eq. 1 as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \tag{2}$$

The solution to Eq. 2 is the equation of motion we've seen before:

$$\theta(t) = \theta_o \cos(\omega t + \phi)$$

where $\omega = \sqrt{g/L}$ and θ_o is the maximum angular displacement.

Ex. 48 A pendulum on Mars. A certain simple pendulum has a period on earth of 1.60 s. What is its period on the surface of Mars, where $g = 3.71\ \mathrm{m/s^2}$.

6 The Physical Pendulum

Most pendulums that oscillate in the vertical plane due to gravity are not as "ideal" as the one described in the previous section. That is, there mass is distributed over a length L and not concentrated at the end of a rigid, massless rod. In order to describe the equation of motion of a physical pendulum, we will appeal to Newton's 2^{nd}, however, we will use it in its rotational form, namely, $\sum \tau = I\alpha$.

Similar to what we did in rotational mechanics, we will calculate the gravitational torque as if all the mass were concentrated at its center of mass. If the distance between the axis of rotation and the center-of-mass is d, we can write Newton's 2^{nd} law as:

$$\sum \tau = I\,\alpha \qquad \Rightarrow \qquad -mg\,d\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{mgd}{I}\right)\sin\theta = 0$$

As before, we will restrict our interest to "small angle" oscillations where we can use the approximation $\sin\theta \approx \theta$. If we do this, then we have an equation similar to the *simple* pendulum:

$$\frac{d^2\theta}{dt^2} + \left(\frac{mgd}{I}\right)\theta = 0$$

where $\omega^2 = mgd/I$, and I is the moment of inertia of the physical pendulum about its axis of rotation.

Ex. 54 We want to support a thin hoop by a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. What must the hoop's radius be?

Also, take a look at problem 73 (not assigned). This is another example of a physical pendulum.

7 Damped Oscillations

As we know, in real life, there are external forces working against oscillating systems, such as friction and air resistance. This results in a less than "ideal" harmonic oscillator solution to the differential equations that arise from applying Newton's 2^{nd} law.

There is one class of external forces where the *friction* is proportional to the velocity, that is, some constant *times* the velocity ($b\,v_x$). Let's assume we have a mass connected to massless spring moving horizontally on a table, but now, we include a friction term such as ($-bv_x$). If we include such a term in Newton's 2^{nd} law, we can write the following:

$$\sum F_x = ma \qquad \Rightarrow \qquad -kx - bv_x = m\frac{d^2x}{dt^2}$$

The solution to this differential equation is

$$x(t) = Ae^{-(b/2m)t}\cos\left(\omega' t + \phi\right) \tag{3}$$

where ω' is:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The amplitude of the motion is:

$$A(t) = Ae^{-bt/2m}$$

So, the equation of motion can be written as:

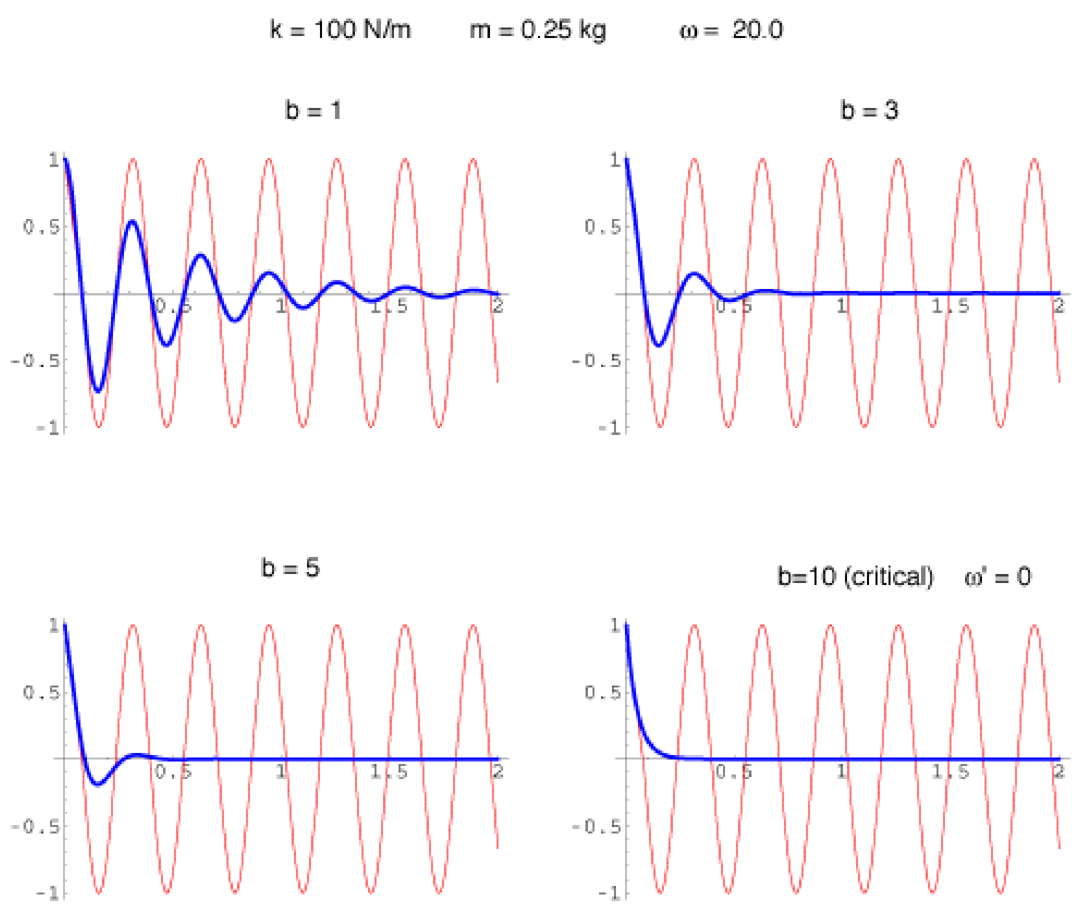
$$x(t) = A(t)\ \cos(\omega' t) \tag{4}$$

Critical Damping occurs when $\omega' = 0$:

$$b = 2\sqrt{km} \qquad \text{(Critical Damping)}$$

Overdamping occurs when ω' is *imaginary* and Eq. 3 (see above) no longer describes the motion of the system:

$$b > 2\sqrt{km} \qquad \text{(Overdamping)}$$



Ex. 61 An unhappy 0.300-kg rodent, moving on the end of a spring with force constant $k = 2.50$ N/m is acted on by a damping force $F_x = -bv_x$. a) If the constant b has the value 0.900 kg/s, what is the frequency of oscillation of the mouse? b) For what value of the constant b will the motion be critically damped?

8 Forced Oscillations and Resonance

A damped oscillator left to itself will eventually come to rest. However, we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic or cyclic way with frequence ω_d. We call this additional force, a **driving force**.

$$F(t) = F_{\mathrm{max}} \cos w_{\mathrm{d}} t$$

The driving frequency ω_d is not the natural oscillating frequency ω', however, the the closer ω_d is to ω', the amplitude of the oscillation can increase dramatically, and create a resonance.

The amplitude of the oscillation as a function of the driving frequency ω_d can be written as

$$A(\omega_d) = \frac{F_{\max}}{\sqrt{\left(k - m\omega_d^2\right)^2 + b^2\omega_d^2}}$$

When the first term in the radical is zero (i.e., $k = m\omega_d^2$), then the amplitude A will reach its maximum near $\omega_d = \sqrt{k/m}$. The height of the curve is proportional to $1/b$.