PREPARED BY: D. Smith
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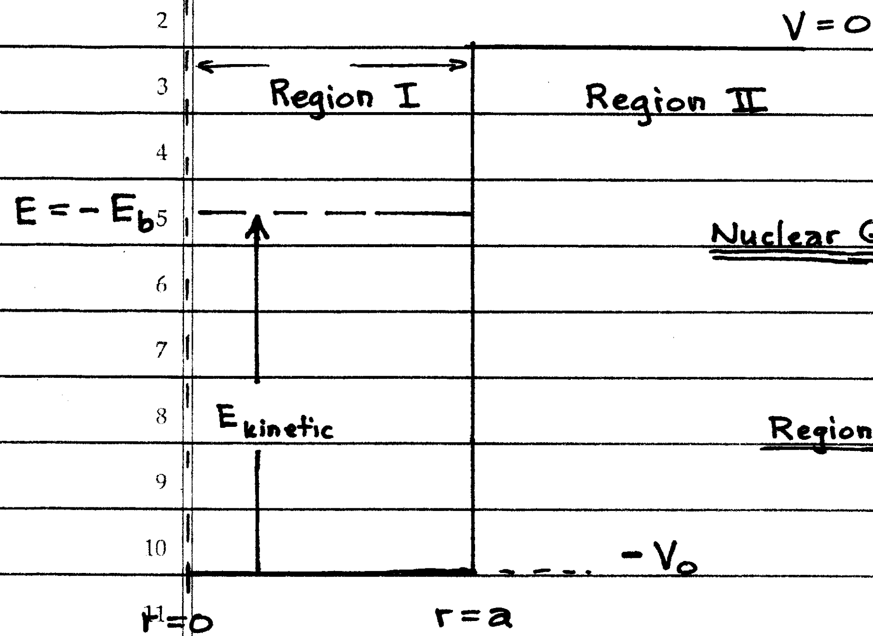
# Finite Spherical Well






**Nuclear Ground State:** $\ell = 0 \quad n = 1$

Radial Equation

**Region I:** $$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right]u = Eu$$

where $u(r) = r\,R(r)$

$V(r) = -V_0 \qquad E = -E_b$

**Region I:** $$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(V_0 - E_b)u = 0$$

Solution: $u(r) = r\,R(r) \Rightarrow A_{n\ell}\, j_\ell(kr)\, r = c_1\, r\, j_0(kr)$

where $\boxed{k = \sqrt{\dfrac{2m(V_0 - E_b)}{\hbar^2}}}$ $\qquad u_I(r) = c_1\, r\, j_0(kr) = \dfrac{c_1}{k}\sin kr$

**Region II:** $$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(-E_b)u = 0$$

Solution: $u_{II}(r) = c_2\, e^{-\kappa r}$ $\qquad$ where $\boxed{\kappa = \sqrt{\dfrac{2mE_b}{\hbar^2}}}$

$$u(r) = r\,R(r) = \begin{cases} \dfrac{c_1}{k}\sin(kr) & 0 < r < a \\ c_2\, e^{-\kappa r} & r > a \end{cases}$$

# Finite Spherical Well



1 Continuity of $(rR)$ $\rightarrow$ $\frac{c_1}{k}\sin ka = c_2 e^{-\kappa a}$ at $r=a$

2 Continuity of $(rR)'$ $\rightarrow$ $c_1 \cos ka = -c_2 \kappa e^{-\kappa a}$ at $r=a$

3 $3^{rd}$ contraint $\Rightarrow$ $\int_0^\infty |u(r)|^2 \, dr = 1$ $u(r)$ must be normalized

Once $V_o$ & $a$ are defined, there are 3 unknowns:
$\Rightarrow$ $c_1$, $c_2$, and $E_b$ ← the energy of the bound state.

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1 ÷ 2 $\quad \frac{1}{k}\frac{\sin ka}{\cos ka} = -\frac{1}{\kappa}$ $\Rightarrow$ (4) $\boxed{\tan ka = -\frac{k}{\kappa}}$

$\tan ka = -\sqrt{\frac{V_o - E_b}{E_b}}$ $\Rightarrow$ $\boxed{\tan\left[\sqrt{\frac{2m}{\hbar^2}(V_o - E_b)}\; a\right] = -\sqrt{\frac{V_o}{E_b} - 1}}$

<u>Assume:</u>

$$\left.\begin{aligned} m &= m_{proton} = 938\ \text{MeV}/c^2 \\ \hbar c &= 197\ \text{MeV}\cdot\text{fm} \\ a &= 4.0\ \text{fm} \\ V_o &= 25\ \text{MeV} \end{aligned}\right\}$$

Using we find that:

$\boxed{E_b = 16.7\ \text{MeV}}$

See the Mathematica attachment

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We divided eq 1 by eq 2 to find the binding energies. Now we need to find a relationship between $c_1$ and $c_2$ and then normalize the wave function $\int_0^\infty |u(r)|^2 \, dr = 1$ to get a unique value for $c_1$ and $c_2$. From eq 1 we have that: $\boxed{c_2 = \frac{c_1}{k}\sin ka \, e^{\kappa a}}$

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# Finite Spherical Well 3

$$u(r) = \begin{cases} c_1\, r\, \dfrac{\sin kr}{kr} & 0 < r \le a \\ c_2\, e^{-\kappa r} & r \ge a \end{cases}$$

$$u(r) = \begin{cases} \dfrac{c_1}{k} \sin kr & 0 < r \le a \\ \dfrac{c_1}{k} \sin ka\; e^{\kappa a}\, e^{-\kappa r} & r \ge a \end{cases}$$

$$u(r) = \frac{c_1}{k} \begin{cases} \sin kr & 0 < r \le a \\ (\sin ka)\, e^{-\kappa(r-a)} & r \ge a \end{cases}$$

What is $\sin ka$? $\quad \sin^2 ka \stackrel{\text{def}}{\equiv} \dfrac{1}{1+\cot^2 ka}$

From Eq. 4 $\quad \cot^2 ka = \dfrac{\kappa^2}{k^2} \qquad$ So, $\sin^2 ka = \dfrac{1}{1+\kappa^2/k^2} = \dfrac{k^2}{k^2+\kappa^2}$

So, $\sin ka = \dfrac{k}{\sqrt{k^2+\kappa^2}} = \sqrt{1-\dfrac{E_b}{V_0}}$ (using def's of $k$ & $\kappa$ from page 1)

So, we have $u(r) = \dfrac{c_1}{k} \begin{cases} \sin kr & 0 < r \le a \\ \sqrt{1-\dfrac{E_b}{V_0}}\; e^{-\kappa(r-a)} & r \ge a \end{cases}$

From the 3rd constraint, we have $\text{Norm} = \displaystyle\int_0^\infty |u(r)|^2\, dr = 1$

$$\text{Norm} = \frac{|c_1|^2}{k^2}\left[\int_0^a \sin^2 kr\, dr + \left(1-\frac{E_b}{V_0}\right)\int_a^\infty e^{-2\kappa(r-a_0)}\, dr\right] = 1$$

Go to the Mathematica program to obtain $|c_1|$ for a given $V_0$ and $E_b$.

# Finding the bound states in a finite spherical well.

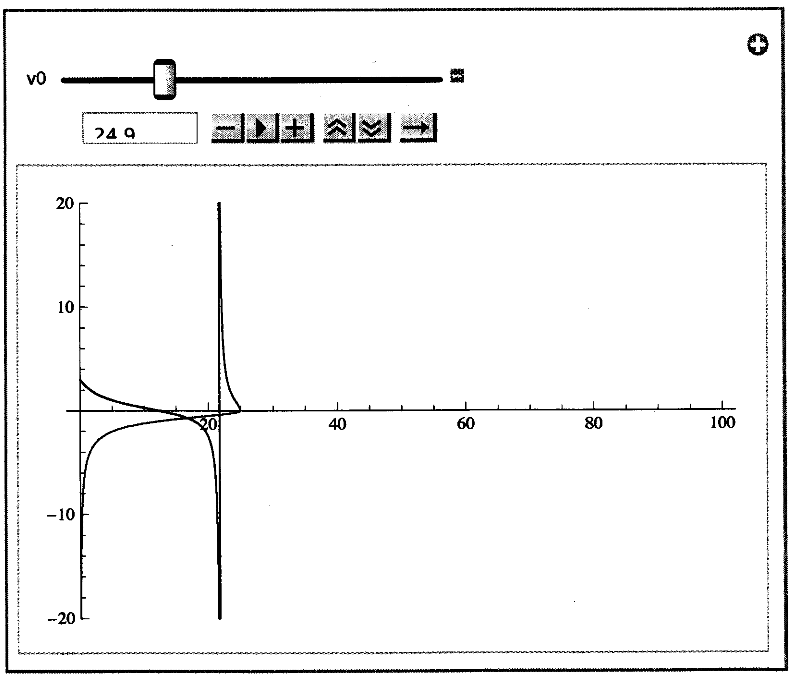
In[546]:=
```
mc2 = 938.;   "mass of a nucleon";
ħc = 197.;
v = 25;       "depth of the potential well";
a = 4.0 ;     "radius of the nucleus" ;
```

# Focussing on the $\ell = 0$ spherical Bessel function, find the minimum potential energy v0 to create the first 3 bound states (i.e., $n$ = 1, 2 and 3).

In[550]:=
```
v0 =.   "Let the depth of the potential well vary from 0 -> 100.";
Eb =.
```

$$\text{Manipulate}\left[\text{Plot}\left[\left\{\text{Tan}\left[\sqrt{\frac{2\ \text{mc2}}{(\hbar c)^2}\ (\text{v0} - \text{Eb})\ a^2}\ \right],\ -\sqrt{\frac{\text{v0}}{\text{Eb}} - 1}\ \right\},\ \{\text{Eb}, 0, 100\},\ \text{PlotRange} \to \{-20, 20\}\right],\ \{\text{v0}, 0, 100\}\right]$$

Out[552]=



- **Provide the depth of the potential energy well "v0" and the starting value for Eb in order to find the intersection point and the value for Eb.**

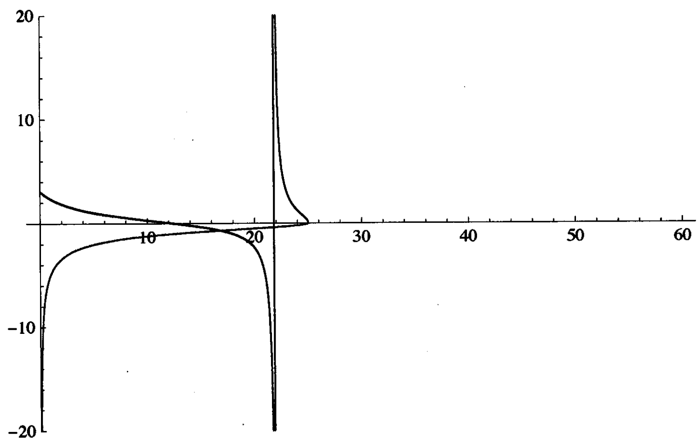
In[928]:=
```
v0 = 25;
Eb =.
EbStart = 2;
```

$$\texttt{Plot}\left[\left\{\texttt{Tan}\left[\sqrt{\frac{2\,\texttt{mc2}}{(\hbar \texttt{c})^2}\,(\texttt{v0}-\texttt{Eb})\,\texttt{a}^2}\right],\ -\sqrt{\frac{\texttt{v0}}{\texttt{Eb}}-1}\right\},\ \{\texttt{Eb},\ 0,\ 60\},\ \texttt{PlotRange}\to\{-20,\ 20\}\right]$$

$$\texttt{FindRoot}\left[\texttt{Tan}\left[\sqrt{\frac{2\,\texttt{mc2}}{(\hbar \texttt{c})^2}\,(\texttt{v0}-\texttt{Eb})\,\texttt{a}^2}\right] == -\sqrt{\frac{\texttt{v0}}{\texttt{Eb}}-1},\ \{\texttt{Eb},\ \texttt{EbStart}\}\right]$$

```
Eb = %[[1, 2]];
```

Out[931]=




Out[932]= {Eb → 16.7314}

## Find the normalization constant $c_1$ and plot the probabilty density $|u(r)|^2$

In[828]:=
```
v0
Eb
u =.
```

Out[828]= 25

Out[829]= 16.7314

In[934]:= $$\mathbf{k} = \sqrt{\frac{2\ \mathbf{mc2}\ (\mathbf{v0} - \mathbf{Eb})}{\hbar\mathbf{c}^2}};$$

$$\kappa = \sqrt{\frac{2\ \mathbf{mc2\ Eb}}{\hbar\mathbf{c}^2}};$$

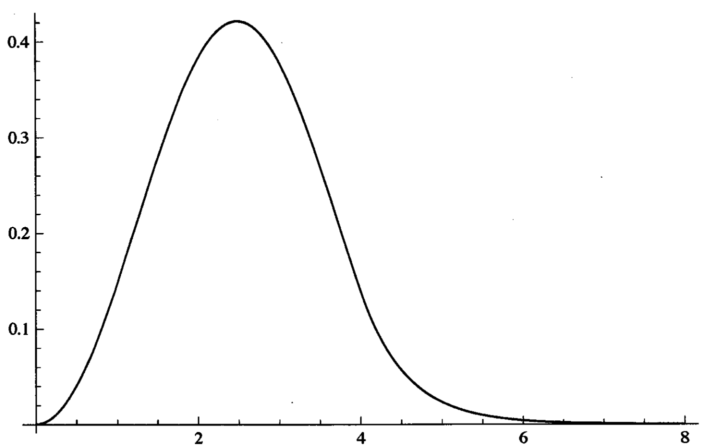
$$\mathbf{const1} = \mathbf{Solve}\left[\frac{\mathbf{c1Sq}}{\mathbf{k}^2}\left(\int_0^a \mathbf{Sin}[\mathbf{k\ r}]^2\, d\mathbf{r} + \mathbf{Sin}[\mathbf{k\ a}]^2 \int_a^\infty e^{-2\kappa\,(\mathbf{r}-\mathbf{a})}\, d\mathbf{r}\right) == 1,\ \mathbf{c1Sq}\right];$$

$$\mathbf{c1} = \sqrt{\mathbf{const1}[[1, 1, 2]]};$$

$$\mathbf{u}[\mathbf{r}] = \begin{cases} \frac{\mathbf{c1}}{\mathbf{k}}\ \mathbf{Sin}[\mathbf{k\ r}] & 0 \le \mathbf{r} \le \mathbf{a} \\ \frac{\mathbf{c1}}{\mathbf{k}}\ \mathbf{Sin}[\mathbf{k\ a}]\ e^{-\kappa\,(\mathbf{r}-\mathbf{a})} & \mathbf{a} \le \mathbf{r} < \infty \end{cases};$$

```
Plot[u[r]^2, {r, 0, 8}]
```

Out[939]=




In[940]:= $$\mathbf{norm} = \int_0^\infty \mathbf{u}[\mathbf{r}]^2\, d\mathbf{r}$$

Out[940]= 1.