Selection Rules (Electric Dipole Transitions)

Matrix Elements

$$\langle \psi_f | \vec{r} | \psi_i \rangle = \langle n'l'm' | \vec{r} | n l m \rangle = \langle n'l'm' | x\hat{i} + y\hat{j} + z\hat{k} | n l m \rangle$$

Using angular momentum commutation relations, we see that:

$$[L_z, x] = i\hbar y \qquad [L_z, y] = -i\hbar x \qquad [L_z, z] = 0$$

$L_z = xp_y - yp_x$

$$0 = \langle n'l'm' | [L_z, z] | n l m \rangle = \langle n'l'm' | L_z z | n l m \rangle - \langle n'l'm' | z L_z | n l m \rangle$$

$$= \langle z L_z n'l'm' | n l m \rangle - \langle n'l'm' | z L_z n l m \rangle$$

$$= m'\hbar \langle n'l'm' | z | n l m \rangle - m\hbar \langle n'l'm' | z | n l m \rangle$$

See Problem 3.5(c)
$(\hat{Q}\hat{P})^{\dagger} = \hat{P}\hat{Q}$
if $\hat{P}$ and $\hat{Q}$ are hermitian

Either $\langle n'l'm' | z | n l m \rangle = 0$ or $(m' - m) = 0$

$$\boxed{\text{If } \Delta m = 0 \text{ then } \langle n'l'm' | z | n l m \rangle \neq 0}$$

$$[L_z, x] = i\hbar y$$

$$\langle n'l'm' | [L_z, x] | n l m \rangle = \langle x L_z n'l'm' | n l m \rangle - \langle n'l'm' | x L_z n l m \rangle$$

$$= m'\hbar \langle n'l'm' | x | n l m \rangle - m\hbar \langle n'l'm' | x | n l m \rangle$$

$$= (m' - m)\hbar \langle n'l'm' | x | n l m \rangle = i\hbar \langle n'l'm' | y | n l m \rangle$$

1 $\Rightarrow \boxed{(m' - m)\hbar \langle n'l'm' | x | n l m \rangle = i\hbar \langle n'l'm' | y | n l m \rangle}$

$$[L_z, y] = -i\hbar x$$

$$\langle n'l'm' | [L_z, y] | n l m \rangle = \langle y L_z n'l'm' | n l m \rangle - \langle n'l'm' | y L_z n l m \rangle$$

Selection Rules (Electric Dipole Transitions)

$$\langle n'\ell'm' | [L_z, y] | n\ell m\rangle = m'\hbar \langle n'\ell'm' | y | n\ell m\rangle - m\hbar \langle n'\ell'm' | y | n\ell m\rangle$$
$$= (m'-m)\hbar \langle n'\ell'm' | y | n\ell m\rangle = -i\hbar \langle n'\ell'm' | x | n\ell m\rangle$$

(2) $\Rightarrow \boxed{(m'-m)\hbar \langle n'\ell'm' | y | n\ell m\rangle = -i\hbar \langle n'\ell'm' | x | n\ell m\rangle}$

Combining (1) and (2) we find:

(1) $\rightarrow \langle n'\ell'm' | y | n\ell m\rangle = -i(m'-m)\langle n'\ell'm' | x | n\ell m\rangle$

(2) $-i(m'-m)^2\langle n'\ell'm' | x | n\ell m\rangle = -i\langle n'\ell'm' | x | n\ell m\rangle$

$$(m'-m)^2 = 1 \quad \Rightarrow \quad m'-m = \pm 1 \qquad \boxed{m' = m \pm 1}$$

In conclusion:

> No electric dipole transitions occur unless: $\Delta m = 0$ or ± 1

Selection Rules Involving ℓ and ℓ'

E.C. Problem 9.12 $\quad [L^2, [L^2, \vec{r}]] = 2\hbar^2 (\vec{r} L^2 + L^2 \vec{r})$

$$\langle n'\ell'm' | [L^2, [L^2, \vec{r}]] | n\ell m\rangle = \langle n'\ell'm' | 2\hbar^2(\vec{r}L^2 + L^2\vec{r}) | n\ell m\rangle$$
$$= 2\hbar^2\left[\langle \vec{r}L^2 n'\ell'm' | n\ell m\rangle + \langle n'\ell'm' | \vec{r}L^2 | n\ell m\rangle\right]$$
$$= 2\hbar^2\left[\ell'(\ell'+1)\hbar^2\langle n'\ell'm' | \vec{r} | n\ell m\rangle + \ell(\ell+1)\hbar^2\langle n'\ell'm' | \vec{r} | n\ell m\rangle\right.$$

$$\boxed{\langle n'\ell'm' | [L^2, [L^2, \vec{r}]] | n\ell m\rangle = 2\hbar^4\left(\ell'(\ell'+1) + \ell(\ell+1)\right)\langle n'\ell'm' | \vec{r} | n\ell m\rangle}$$

Using the Problem 9.12 substitution

Selection Rules (Electric Dipole Transition)

Calculate Again ..... w/o the Problem 9.12 substitution

$$\langle n'\ell' m' | [L^2, [L^2, r]] | n\ell m\rangle = \langle n'\ell' m' | L^2[L^2,\vec{r}] - [L^2,\vec{r}]L^2 | n\ell m\rangle$$
$$= \langle n'\ell' m' | L^2L^2\vec{r} - L^2\vec{r}L^2 - L^2\vec{r}L^2 + \vec{r}L^2L^2 | n\ell m\rangle$$
$$= \langle n'\ell' m' | L^4\vec{r} - 2L^2\vec{r}L^2 + \vec{r}L^4 | n\ell m\rangle$$
$$= [\ell'(\ell'+1)]^2\hbar^4\langle n'\ell' m' | \vec{r} | n\ell m\rangle - 2\ell'(\ell'+1)\ell(\ell+1)\hbar^4\langle n'\ell' m' | \vec{r} | n\ell m\rangle$$
$$+ [\ell(\ell+1)]^2\hbar^4 \langle n'\ell' m' | \vec{r} | n\ell m\rangle$$

$$\boxed{= [\ell'(\ell'+1) - \ell(\ell+1)]^2 \hbar^4 \langle n'\ell' m' | \vec{r} | n\ell m\rangle}$$

w/o the Prob. 9.12 substitution

Setting the 2 answers equal to each other we obtain:

$$2\hbar^4(\ell'(\ell'+1) + \ell(\ell-1)) = [\ell'(\ell'+1) - \ell(\ell+1)]^2\hbar^4$$

$$\Rightarrow [\ell'(\ell'+1) - \ell(\ell+1)]^2 - 2(\ell'(\ell'+1) + \ell(\ell+1)) = 0$$

However: $\ell'(\ell'+1) - \ell(\ell+1) = (\ell'+\ell+1)(\ell'-\ell)$

So: $[(\ell'+\ell+1)(\ell'-\ell)]^2 - 2(\ell'(\ell'+1) + \ell(\ell+1)) = 0$

$$\Rightarrow \underbrace{[(\ell'+\ell+1)^2 - 1]}_{①}\underbrace{[(\ell'-\ell)^2 - 1]}_{②} = 0$$

← This factorization is not obvious .... but it's true!! Work it out.

① not equal to zero unless $\ell' = \ell = 0$ See problem 9.13

② $\ell' - \ell = \pm 1$ or $\ell' = \ell \pm 1$

Thus, the selection rule for ℓ is: $\boxed{\Delta\ell = \pm 1}$ for electric dipole transitions.

N.B. Photon carries of $1\hbar$ of spin angular momentum.