Quantum Physics by Robert Eisberg & Robert Resnick 2nd Ed. (1985)

Appendix O
THE THOMAS PRECESSION

The relativistic effect which introduces the factor of 1/2 in (8-25) for the spin-orbit orientational potential energy is called the Thomas precession. It is not difficult to understand if we keep the geometry sufficiently simple. For this purpose, let us assume that the electron moves about the nucleus in a circular Bohr orbit, as illustrated in Figure O-1. The figure shows the situation as seen by an observer in the nuclear rest frame xy. The electron is momentarily at rest in the frame x_1y_1 at the instant t_1, and momentarily at rest in the frame x_2y_2 at the slightly later instant t_2. Both the axes of xy and of x_2y_2 have been constructed parallel to the axes of x_1y_1, as seen by an observer in x_1y_1. Nevertheless, we shall show that the observer in xy sees the axes of x_2y_2 rotated slightly relative to his own axes. He sees the axes of the x_3y_3 frame rotated even more, etc. Thus he sees that the set of axes in which the electron is instantaneously at rest are precessing, relative to his own set of axes, as the electron goes around the nucleus—even though the observers instantaneously at rest relative to the electron contend that each set of axes $x_{n+1}y_{n+1}$ is parallel to the preceding set x_ny_n. By using a sequence of reference frames x_ny_n in which the electron is momentarily at rest, and which are each moving with constant velocity relative to the others and relative to the xy frame, we can apply special relativity theory to the problem even though the electron is accelerating relative to the xy frame.

Figure O-2 shows xy, x_1y_1, and x_2y_2 from the point of view of the observer in x_1y_1. Since the electron is moving with velocity $\mathbf{v}$ relative to the nucleus, the axes xy are moving with velocity $-\mathbf{v}$ in the direction of the negative x_1 axis relative to x_1y_1. As seen in x_1y_1, the electron is accelerating toward the nucleus with acceleration $\mathbf{a}$ in the direction of the positive y_1 axis. If the time interval $(t_2 - t_1)$ is very small, the change in velocity of the electron in that interval is

$$d\mathbf{v} = \mathbf{a}(t_2 - t_1) = \mathbf{a}\,dt \tag{O-1}$$

and this will be the velocity of x_2y_2 as seen by x_1y_1. Now let us use the relativistic velocity transformation equations of Appendix A to evaluate the components of $\mathbf{u}_a$, the velocity of x_2y_2 as seen by xy. These give

$$u_{a_x} = \frac{dv_x - v_x}{1 - \dfrac{v_x dv_x}{c^2}} = \frac{0 + v}{1 - \dfrac{-v \cdot 0}{c^2}} = v$$

$$u_{a_y} = \frac{dv_y\sqrt{1 - \dfrac{v_x^2}{c^2}}}{1 - \dfrac{v_x dv_x}{c^2}} = dv\sqrt{1 - \frac{v^2}{c^2}}$$

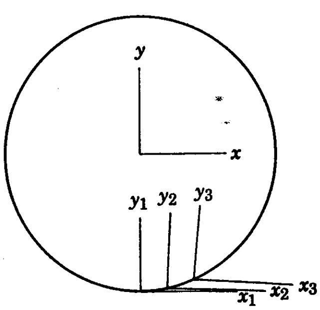

Figure O-1 The frames of reference used in calculating the Thomas precession.

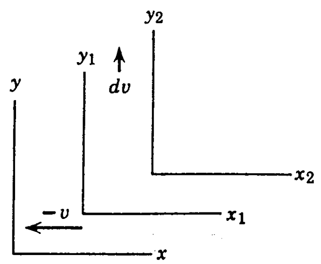

Figure O-2 The frames of reference used in calculating the Thomas precession, as seen in the x_1y_1 frame.

Using the same transformation equations to evaluate the components of $\mathbf{u}_b$, the velocity of xy as seen by x_2y_2, we have

$$u_{b_x} = \frac{v_x\sqrt{1-\dfrac{dv_y^2}{c^2}}}{1-\dfrac{dv_yv_y}{c^2}} = \frac{-v\sqrt{1-\dfrac{dv^2}{c^2}}}{1-\dfrac{dv\cdot 0}{c^2}} = -v\sqrt{1-\frac{dv^2}{c^2}}$$

$$u_{b_y} = \frac{v_y - dv_y}{1-\dfrac{dv_yv_y}{c^2}} = -dv$$

Next we calculate the angle between the vector $\mathbf{u}_a$ and the x axis of the xy frame. It is

$$\theta_a = \frac{u_{a_y}}{u_{a_x}} = \frac{dv\sqrt{1-\dfrac{v^2}{c^2}}}{v}$$

The angle between the vector $\mathbf{u}_b$ and the x axis of the x_2y_2 frame is

$$\theta_b = \frac{u_{b_y}}{u_{b_x}} = \frac{-dv}{-v\sqrt{1-\dfrac{dv^2}{c^2}}}$$

Figure O-3 shows the x_2y_2 and xy frames from the point of view of xy. Because of the equivalence of inertial frames, $\mathbf{u}_a$ and $\mathbf{u}_b$ must be exactly opposite in direction. Since the angles between the x axes and the relative velocity vectors are not the same, the x_2y_2 frame appears to be rotated relative to the xy frame. The angle of rotation is

$$d\theta = \theta_b - \theta_a = \left(\frac{dv}{v\sqrt{1-\dfrac{dv^2}{c^2}}} - \frac{dv}{v}\sqrt{1-\frac{v^2}{c^2}}\right)$$

As dv is a differential, we may neglect dv^2/c^2 and obtain

$$d\theta = \frac{dv}{v}\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$$

As the velocity of an electron in a one-electron atom is relatively small compared to the velocity of light, $v^2/c^2 \ll 1$. (This is also true for the electrons responsible for the optical spectra in other atoms.) Thus we may obtain an excellent approximation to $d\theta$ by making a binomial

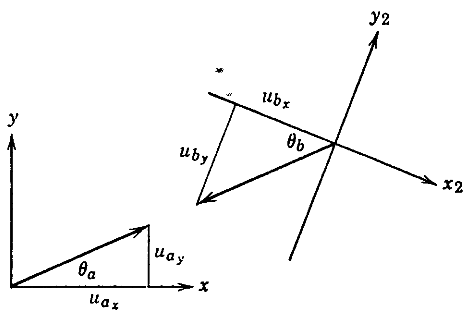

Figure O-3 An exaggerated illustration of the Thomas precession.

expansion of the square root, keeping only the first two terms. That is

$$d\theta \simeq \frac{dv}{v}\left[1-\left(1-\frac{v^2}{2c^2}\right)\right]$$

$$=\frac{dv\,v^2}{2vc^2}=\frac{v\,dv}{2c^2}=\frac{va\,dt}{2c^2}$$

where we have evaluated dv from (O-1). The axes in which the electron is instantaneously at rest appear to precess, relative to the nucleus, with the so-called *Thomas frequency*

$$\omega_T=\frac{d\theta}{dt}=\frac{va}{2c^2}$$

Inspection of the figures will verify that the sense of precession is given by the vector equation

$$\boldsymbol{\omega}_T=-\frac{1}{2c^2}\,\mathbf{v}\times\mathbf{a} \tag{O-2}$$

Relative to frames in which the electron is at rest, its spin magnetic dipole moment precesses in the magnetic field it experiences at the Larmor frequency $\boldsymbol{\omega}$. But these frames are themselves precessing with frequency $\boldsymbol{\omega}_T$ relative to the frame in which the nucleus is at rest. Consequently, the dipole moment is seen in the nuclear rest frame to precess with angular frequency

$$\boldsymbol{\omega}'=\boldsymbol{\omega}+\boldsymbol{\omega}_T \tag{O-3}$$

Using an equation analogous to (8-14), plus (8-24), and evaluating g_s and μ_b, we have

$$\omega=-\frac{g_s\mu_b}{c^2\hbar}\,\mathbf{v}\times\mathbf{E}=-\frac{2e\hbar}{2mc^2\hbar}\,\mathbf{v}\times\mathbf{E}=-\frac{e}{mc^2}\,\mathbf{v}\times\mathbf{E} \tag{O-4}$$

To evaluate $\boldsymbol{\omega}_T$ in similar terms, we may use Newton's law to express the acceleration of the electron as a function of the electric field: $\mathbf{a}=\mathbf{F}/m=-e\mathbf{E}/m$. With this, (O-2) yields

$$\boldsymbol{\omega}_T=\frac{e}{2mc^2}\,\mathbf{v}\times\mathbf{E} \tag{O-5}$$

Thus, the precessional frequency in the nuclear rest frame is

$$\boldsymbol{\omega}'=-\frac{e}{mc^2}\,\mathbf{v}\times\mathbf{E}+\frac{e}{2mc^2}\,\mathbf{v}\times\mathbf{E}=-\frac{e}{2mc^2}\,\mathbf{v}\times\mathbf{E} \tag{O-6}$$

Comparing (O-4) and (O-6), we see that the effect of transforming the spin magnetic dipole precession frequency, from the frames in which the electron is at rest to the normal frame in which the nucleus is at rest, is to reduce its magnitude by exactly a factor of 1/2. The same is true of the orientational potential energy ΔE since the magnitude of that quantity is proportional to the magnitude of the precession frequency ω. This can be seen from equations analogous to (8-13) and (8-14)

$$\Delta E=-\boldsymbol{\mu}_s\cdot\mathbf{B}=\frac{g_s\mu_b}{\hbar}\,\mathbf{S}\cdot\mathbf{B}$$

and

$$\boldsymbol{\omega}=\frac{g_s\mu_b}{\hbar}\,\mathbf{B}$$

Thus we have completed our verification of the factor of 1/2 in (8-25).

PROBLEM

1. The Thomas precession can also be described in terms of a time dilation between the reference frame in which the nucleus is at rest and the reference frames in which the electron is instantaneously at rest, which leads to a disagreement between an observer at the nucleus and the observers at the electron concerning the time required for each to make a complete revolution about the other. Work out the details of this description, and compare with the results of Appendix O.