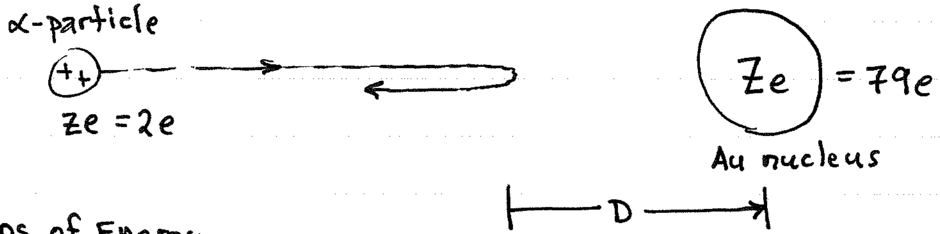
Rutherford Scattering

1. Alpha particle scattering off a thin gold foil.
   What is the distance of closest approach ?



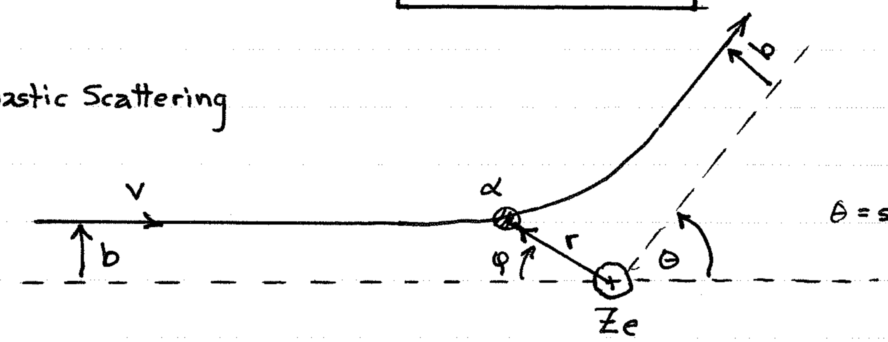
Cons. of Energy

$$K = \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{D}$$

$$\boxed{D = \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{K}}$$

D = distance of closest approach

Elastic Scattering



θ = scattering angle

$\vec{L} = \vec{r} \times \vec{p}$ constant

$L = mvb$ $\quad\quad$ $L = I\omega = mr^2 \dfrac{d\varphi}{dt}$

Find $r(\varphi)$

Newton's 2^{nd} Law: $\quad \sum F = ma_r$

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 \quad\quad \frac{1}{4\pi\varepsilon_0}\frac{Zze^2}{r^2} = m\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right]$$

$$r = \frac{1}{u} \quad\quad \boxed{\frac{d\varphi}{dt} = \frac{L}{mr^2}} \quad\quad \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\varphi}\frac{d\varphi}{dt} = \frac{-1}{u^2}\frac{du}{d\varphi}\frac{L}{m}u^2$$

Rutherford Scattering

$$\frac{dr}{dt} = -\frac{L}{m}\frac{du}{d\varphi}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\varphi}\left(\frac{dr}{dt}\right)\frac{d\varphi}{dt} = -\frac{L}{m}\frac{d^2u}{d\varphi^2}\frac{d\varphi}{dt} = \frac{-L}{m}\frac{d^2u}{d\varphi^2}\left(\frac{L}{mr^2}\right)$$

$$\frac{d^2r}{dt^2} = \frac{-L^2}{m^2r^2}\frac{d^2u}{d\varphi^2}$$

$$\Sigma F = ma \Rightarrow \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{r^2} = m\left[\frac{-L^2}{m^2r^2}\frac{d^2u}{d\varphi^2} - r\left(\frac{L^2}{m^2r^4}\right)\right]$$

$$\frac{Zze^2}{4\pi\varepsilon_0}u^2 = \frac{-L^2u^2}{m}\frac{d^2u}{d\varphi^2} - \frac{L^2}{m}u^3$$

$$\Rightarrow \frac{d^2u}{d\varphi^2} + u = -\frac{m}{L^2}\frac{Zze^2}{4\pi\varepsilon_0} = -\frac{DK\,m}{(mvb)^2} = \frac{-D\left(\frac{1}{2}mv^2\right)m}{m^2v^2b^2}$$

$$\boxed{\frac{d^2u}{d\varphi^2} + u = -\frac{D}{2b^2}}$$

Solution to this equation: $u(\varphi) = A\sin\varphi + B\cos\varphi - \frac{D}{2b^2}$

Initial Conditions: (1) $u = 0$ when $\varphi = 0$

$v = -\frac{dr}{dt} = +\frac{L}{m}\frac{du}{d\varphi}$ (2) $\frac{du}{d\varphi} = \frac{mv}{L} = \frac{1}{b}$ when $\varphi = 0$

(1) $u(0) = A\sin 0 + B\cos 0 - D/2b^2 \Rightarrow \boxed{B = D/2b^2}$

(2) $\frac{du}{d\varphi} = A\cos\varphi - B\sin\varphi \Rightarrow \left.\frac{du}{d\varphi}\right|_{\varphi=0} = \frac{1}{b} = A\cos 0 - B\sin 0$

$\Rightarrow \boxed{A = \frac{1}{b}}$

Rutherford Scattering

NOTES

$$u(\varphi) = \frac{1}{b}\sin\varphi + \frac{D}{2b^2}\cos\varphi - \frac{D}{2b^2} = \frac{1}{b}\sin\varphi - \frac{D}{2b^2}\left(1-\cos\varphi\right)$$

$$u(\varphi) = \frac{1}{b}\sin\varphi - \frac{D}{2b^2}\left(1-\cos\varphi\right) \Rightarrow \boxed{\frac{1}{r} = \frac{1}{b}\sin\varphi - \frac{D}{2b^2}\left(1-\cos\varphi\right)}$$

After the scatter $r \to \infty \Rightarrow \varphi \to \pi - \theta$

$$\frac{1}{\infty} = \frac{1}{b}\sin(\pi-\theta) - \frac{D}{2b^2}\left(1-\cos(\pi-\theta)\right) \Rightarrow 0 = \frac{1}{b}\sin\theta - \frac{D}{2b^2}\left(1+\cos\theta\right)$$

$$0 = \frac{2\sin\theta/2\,\cos\theta/2}{b} - \frac{D}{2b^2}\,2\cos^2\theta/2$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{D}{2b}\cos\frac{\theta}{2} \Rightarrow \boxed{\cot\frac{\theta}{2} = \frac{2b}{D}}$$

The relation betweeen θ and b.

Let's take a look at the target:

① $$\frac{\#\text{ of nuclei}}{m^3} = \frac{\rho N_A}{m} \left[\frac{gm/m^3 \;\; \#/mole}{gm/mole}\right] \Rightarrow \left[\frac{\#}{m^3}\right]$$

Atomic mass

② $n = \#$ of scatterers $\Rightarrow n = \frac{\rho N_A}{m} A_o \delta$

thickness of target

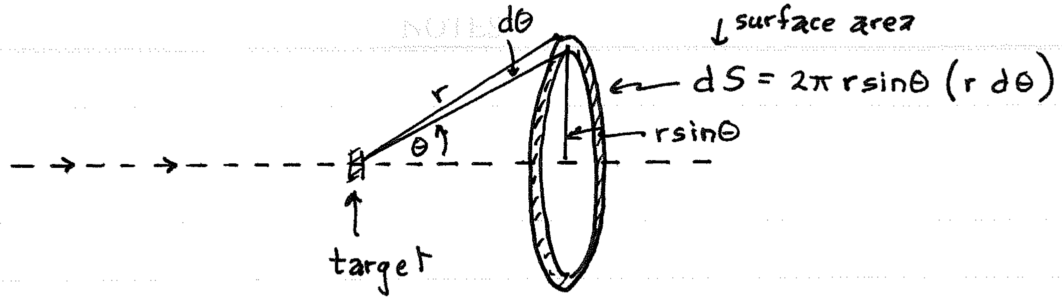
cross-sectional area of the beam

Beam Intensity $= I_o \left[\frac{\text{particles}}{\text{time}\cdot\text{Area}}\right]$

flux intensity

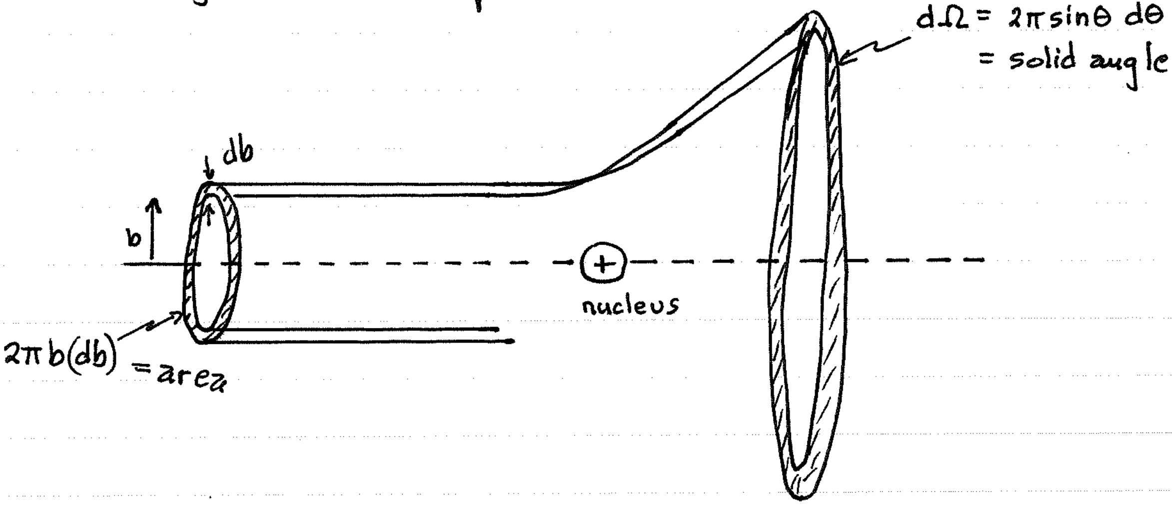
$$N_o = I_o A_o = \left[\frac{\#\text{ of particles}}{\text{second}}\right]$$

Rutherford Scattering



differential surface area $= 2\pi r^2 \sin\theta \, d\theta$

solid angle $= d\Omega \equiv \dfrac{dS}{r^2} = 2\pi \sin\theta \, d\theta$



$$\frac{\text{\# of particles incident}}{\text{time}} = 2\pi b \, (db) \, I_o$$

$$\frac{\text{\# of particles scattered into } d\Omega}{\text{time (single nucleus)}} = \frac{dN}{n}$$

dN ← # of particles/sec

n ← # of scatterers

$$\frac{dN}{n} = 2\pi b \, (db) \, I_o \quad \Rightarrow \quad \boxed{d\sigma \equiv \frac{dN}{n I_o} = 2\pi b \, (db)}$$

$$d\sigma = \frac{dN}{n \, N_o / A_o} = \frac{dN / N_o}{n / A_o} = \frac{\dfrac{\text{\# of particles} \to d\Omega / \text{time}}{\text{\# of particles / time}}}{\dfrac{\text{\# of target nuclei encountered}}{\text{area of the beam}}}$$

cross-sectional area of a single nucleus ⟶ $d\sigma = \dfrac{\text{\# of particles} \to d\Omega \text{ by one nucleus}}{\text{\# of incident particles per unit area of beam}}$

Rutherford Scattering

NOTES

$$1 \text{ barn} \equiv 10^{-28}\, m^2 = 10^{-24}\, cm^2 = (10^{-12}\, cm)^2$$

$$\boxed{d\sigma = 2\pi b\,(db)}$$ Find db in terms of $d\theta \to d\Omega$

$$\cot\frac{\theta}{2} = \frac{2b}{D} \qquad -\csc^2\theta/2\ \frac{d\theta}{2} = \frac{2}{D}\, db$$

$$\boxed{db = -\frac{D}{4}\csc^2\frac{\theta}{2}\, d\theta}$$

$$d\sigma = 2\pi\left(\frac{D}{2}\cot\frac{\theta}{2}\right)\left(\frac{D}{4}\csc^2\frac{\theta}{2}(-d\theta)\right)$$

$$d\sigma = \frac{\pi D^2}{4}\,\frac{\cos\theta/2}{\sin\theta/2}\,\frac{1}{\sin^2\theta/2}\,|d\theta| = \frac{\pi D^2}{8}\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^4\theta/2}\,|d\theta|$$

$$d\sigma = \frac{D^2}{16}\,\frac{2\pi\sin\theta\, d\theta}{\sin^4\frac{\theta}{2}} = \frac{D^2}{16}\,\frac{d\Omega}{\sin^4\frac{\theta}{2}} = \frac{1}{16}\left(\frac{Zze^2}{4\pi\varepsilon_0 K}\right)^2\frac{d\Omega}{\sin^4\frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\,\frac{Z^2z^2}{K^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2\frac{(\hbar c)^2}{\sin^4\frac{\theta}{2}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{16}\,\frac{Z^2z^2}{K^2}\,\frac{\alpha^2(\hbar c)^2}{\sin^4\frac{\theta}{2}}}$$ Differential Scattering Cross Section for Rutherford Scattering.

Problem: Consider a beam of 5.30 MeV α particles incident upon a thin gold foil.

# of nuclear scatterers

$$n \equiv \frac{\#\text{ of atomic nuclei}}{m^3} \overset{\times A_0\delta}{=} \frac{\rho N_A}{m} \overset{\times A_0\delta}{=} \frac{(19.3\times10^3\, kg/m^3)}{0.197\, kg/mole}\,(6.02\times10^{23}\ \text{nuclei/mole}) \times A_0\,\delta$$

At. mass $\to m$

Rutherford Scattering

$n = 5.90 \times 10^{28}\ \text{nuclei}/m^3\ (A_o\ \delta)$

$$\frac{n}{A_o} = 5.90 \times 10^{28}\ \frac{nuc.}{m^3}\ (\overset{\delta}{\overset{\downarrow}{2.10 \times 10^{-7}\,m}})$$

$$\boxed{\frac{n}{A_o} = 1.24 \times 10^{22}\ \frac{\text{nuclei}}{m^2}}$$

$$dN = \underset{\text{unit time}}{\underline{\#\text{ of } \alpha \text{ particles scattered}}} \to d\Omega \quad = \overset{N_o/A_o}{\overset{\swarrow}{n\, I_o}}\, d\sigma$$

$$dN = n\,\frac{N_o}{A_o}\, d\sigma \qquad dN = \left(\frac{n}{A_o}\right) N_o\, d\sigma$$

Question: If $N_o = 10^4 \left(\frac{\alpha \text{ particles}}{\text{sec}}\right)$ how many particles are scattered into the backward hemisphere?

Answer:

dN particles are scattered into the backward hemisphere/sec.

Question: What is the cross-section (σ) for scattering α particles into the backward hemisphere?

$$\sigma(\pi/2 \to \pi) = \int_{\pi/2}^{\pi} \frac{d\sigma}{d\Omega}\, d\Omega = \frac{1}{16}\,\frac{Z^2 z^2}{K^2} \int_{\pi/2}^{\pi} \frac{\alpha^2 (\hbar c)^2}{\sin^4\frac{\theta}{2}}\, 2\pi \sin\theta\, d\theta$$

$$\sigma(\pi/2 \to \pi) = \frac{Z^2 z^2}{16 K^2}\,\alpha^2 (\hbar c)^2\, 2\pi \underbrace{\int_{\pi/2}^{\pi} \frac{\sin\theta\, d\theta}{\sin^4\frac{\theta}{2}}}_{=2}$$

$$\sigma(\pi/2 \to \pi) = \frac{(79)^2 (2)^2 \left(\frac{1}{137}\right)^2 (197\ \text{MeV-fm})^2\, 2\pi\, (2)}{16\,(5.30\ \text{MeV})^2} = 1443 \times 10^{-30}\, m^2$$

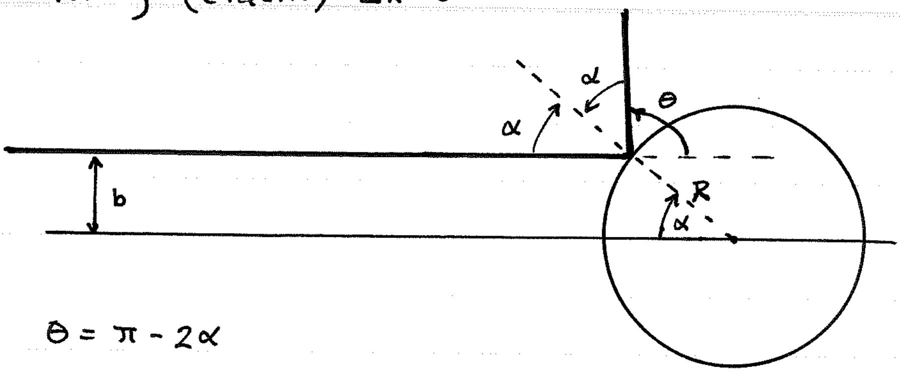
$$\boxed{\sigma(\text{backwards}) = 14.43\ \text{barns}}$$

$$\Delta N = \left(\frac{n}{A_o}\right) N_o\, \sigma = 1.24 \times 10^{22}\ \frac{\text{nucl.}}{m^2}\ 10^4\ \frac{\alpha}{\text{sec}}\ (1443 \times 10^{-30}\, m^2)$$

into the backward hemisphere → $\boxed{\Delta N = 0.179\ \frac{\alpha \text{ particles}}{\text{sec}}}$

Scattering

Hard-Sphere scattering (elastic) $\Delta K = 0$



Example 11.1 $\theta = \pi - 2\alpha$

impact parameter $b \equiv R \sin\alpha$ $\alpha = \frac{\pi - \theta}{2}$

$$b = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right)$$

Recall from Rutherford Scattering: $d\sigma = 2\pi b \, db$

For hard sphere scattering $db = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)d\theta$

$$d\sigma = 2\pi\left(R\cos\left(\frac{\theta}{2}\right)\right)\left(\frac{-R}{2}\sin\left(\frac{\theta}{2}\right)d\theta\right) \xrightarrow{\text{abs. value}} = \frac{R^2}{4}\underbrace{\left(2\pi\sin\theta\, d\theta\right)}_{d\Omega} = \frac{R^2}{4}d\Omega$$

So, $\boxed{\frac{d\sigma}{d\Omega} = \frac{R^2}{4}}$ for hard-sphere scattering

Total Cross Section for hard-sphere scattering:

$$\sigma_{TOT} = \int \frac{d\sigma}{d\Omega}d\Omega = \frac{R^2}{4}2\pi\int_0^\pi \sin\theta\, d\theta = \frac{\pi R^2}{2}(-\cos\theta)\Big|_0^\pi = \underbrace{\pi R^2}_{\text{area of a circle}}$$