Notation (A few words)

Introducing the Dirac "bra" "ket" notation: "bra" = $\langle\ |$

Assuming an infinitely deep potential well "ket" = $|\ \rangle$

An eigenfunction $\psi_n(x)$ can be represented in this short-hand notation

$$\psi_n(x) \rightarrow |n\rangle \qquad |n\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

An integral $\Rightarrow$ $\langle n|n\rangle = \int_0^L \psi_n^*(x)\,\psi_n(x)\,dx = 1$

$$\langle m|n\rangle = \int_0^L \psi_m^*(x)\,\psi_n(x)\,dx = \underset{\text{Kronecker delta}}{\delta_{m,n}} \begin{cases} = 1 & \text{if } m = n \\ = 0 & \text{if } m \neq n \end{cases}$$

Notice that $\langle 1|1\rangle = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx = 1$

and $\langle 2|1\rangle = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx = 0$

$\boxed{|n\rangle = \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \text{ is a set of orthonormal wave functions.}}$

Eigenvalue Equation

$$\underset{\text{energy operator}}{E}\ \underset{\text{eigenfunction}}{|1\rangle} = \underset{\text{eigenvalue}}{\left(\frac{1^2\pi^2\hbar^2}{2mL^2}\right)}|1\rangle = E_0\,|1\rangle$$

where $E_0 = \dfrac{\pi^2\hbar^2}{2mL^2}$

$$\underset{\text{Energy Operator}}{E}\ |n\rangle = \underset{\text{eigenvalue}}{\left(\frac{n^2\pi^2\hbar^2}{2mL^2}\right)}|n\rangle$$

$|n\rangle$ = eigenfunction

where $n^2\left(\dfrac{\pi^2\hbar^2}{2mL^2}\right) = n^2 E_0$

Expectation Values

$\underline{n=1}$ $\qquad \bar{x} = \langle x\rangle = \langle 1|x|1\rangle = \frac{2}{L}\int_0^L x \sin^2\frac{\pi x}{L}\,dx = \frac{L}{2}$

Notation (A few words)

$$\overline{x^2} = \langle x^2 \rangle = \langle 1 | x^2 | 1 \rangle = \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{L^2}{3} - \frac{L^2}{2\pi^2}$$

$$\overline{E} = \langle E \rangle = \langle 1 | \underbrace{E | 1 \rangle}_{= E_1 |1\rangle} = \langle 1 | E_1 | 1 \rangle = E_1 \langle 1 | 1 \rangle = E_1$$

Expectations for any "n"

$$\overline{x} = \langle x \rangle_n = \langle n | x | n \rangle = \frac{2}{L}\int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}$$

$$\overline{x^2} = \langle x^2 \rangle_n = \langle n | x^2 | n \rangle = \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L^3}{3} - \frac{1}{2n^2\pi^2}$$

Infinitely Deep 2-dimensional well

Eigenfunction: $\psi_{n_x, n_y}(x,y) = \frac{2}{L}\left(\sin \frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) = |n_x, n_y\rangle$

$n_x = 1, 2, 3, \ldots$

$n_y = 1, 2, 3, \ldots$

$|n_x, n_y\rangle$ represent a complete set of orthonormal wave functions

$$\langle 1,1 | 1,1 \rangle = \left(\frac{2}{L}\right)^2 \int_0^L dy \int_0^L dx \, \sin^2\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{\pi y}{L}\right) = 1$$

$$\langle 2,1 | 1,1 \rangle = \left(\frac{2}{L}\right)^2 \int_0^L dy \int_0^L dx \, \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{\pi y}{L}\right) = 0$$

Because $\langle 2 | 1 \rangle = 0$ "x" integration

$$\underset{\text{Energy Operator}}{E} |n_x, n_y\rangle = \overbrace{(n_x^2 + n_y^2)\frac{\pi^2 \hbar^2}{2mL^2}}^{\text{Energy Eigenvalue}} |n_x, n_y\rangle = \underbrace{(n_x^2 + n_y^2) E_o}_{\text{Energy Eigenvalue}} |n_x, n_y\rangle$$

where $\boxed{E_o = \frac{\pi^2 \hbar^2}{2mL^2}}$

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Notation (A few words)

Average energy

$$\langle E\rangle = \langle n_x, n_y | E | n_x, n_y\rangle$$

$$= \langle n_x, n_y | \left(\frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2)\right) | n_x, n_y\rangle$$

$$= (n_x^2+n_y^2)E_o \langle n_x, n_y | n_x, n_y\rangle = (n_x^2+n_y^2)E_o \underbrace{\delta_{n_x,n_x}}_{=1}\underbrace{\delta_{n_y,n_y}}_{=1}$$

$$\langle E\rangle = (n_x^2+n_y^2)E_o$$

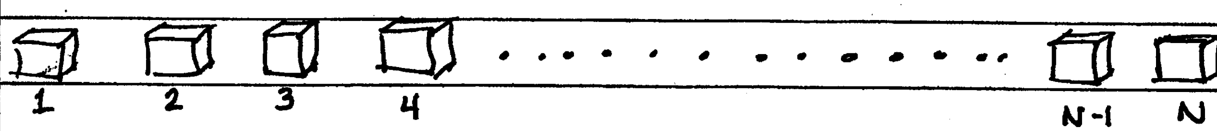
$$\overline{x^2} = \langle x^2\rangle = \langle n_x, n_y | x^2 | n_x, n_y\rangle = \langle n_x | x^2 | n_x\rangle \underbrace{\langle n_y | n_y\rangle}_{=1}$$

$$\overline{x^2} = \langle x^2\rangle = \frac{2}{L}\int_0^L dx \sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_x\pi x}{L}\right)x^2$$

$$\boxed{\langle x^2\rangle = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2 n_x^2}\right)}$$

Example: An ensemble of quantum systems is described by the following wave function: Assume an infinitely deep potential well.

$$\psi = A\,(2|1\rangle + 3|2\rangle + 1|4\rangle)$$



where N = large

a.) Normalize this wave function.

$$\psi^*\psi = (\langle 1|2 + \langle 2|3 + \langle 4|1)(2|1\rangle + 3|2\rangle + 1|4\rangle)$$

$$= A^2\left(4\underbrace{\langle 1|1\rangle}_{=1} + 9\underbrace{\langle 2|2\rangle}_{=1} + 1\underbrace{\langle 4|4\rangle}_{=1}\right) = 14A^2 = 1 \qquad A = \frac{1}{\sqrt{14}}.$$

b.) What is the probability that you will measure the energy of a system to be E_2? $\psi = \frac{1}{\sqrt{14}}(2|1\rangle + 3|2\rangle + 1|4\rangle)$

$$\psi^*\psi = \frac{4}{14}\langle 1|1\rangle + \frac{9}{14}\langle 2|2\rangle + \frac{1}{14}\langle 4|4\rangle \qquad \boxed{\text{Answer} = \frac{9}{14}}$$