

# Chapter 9

## Rotation of Rigid Bodies

### 1 Angular Velocity and Acceleration

$$\theta = \frac{s}{r} \quad (\text{angular displacement})$$

The *natural* units of  $\theta$  is radians.

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

#### Angular Velocity

Usually we pick the  $z$ -axis as the direction about which the rigid body rotates.

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \quad (\text{average angular velocity})$$

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad (\text{definition of angular velocity})$$

The angular velocity can also be written as a vector. Its magnitude is defined by the above equation while its direction is defined by the *right-hand* rule.

#### Angular Acceleration

The average angular acceleration is defined by:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \quad (\text{average angular acceleration})$$

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad (\text{definition of angular acceleration})$$

$$\omega = \frac{d\theta}{dt} \quad \text{and} \quad \alpha = \frac{d\omega}{dt} = \frac{d}{dt} \frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$

**Ex. 4** A fan blade rotates with angular velocity given by  $\omega_z(t) = \gamma - \beta t^2$ , where  $\gamma = 5.00$  rad/s and  $\beta = 0.800$  rad/s<sup>3</sup>. a) Calculate the angular acceleration as a function of time. b) Calculate the instantaneous angular acceleration  $\alpha_z$  at  $t = 3.00$  s and the average angular acceleration  $\alpha_{\text{av-z}}$  for the time interval  $t = 0$  to  $t = 3.00$  s. How do these two quantities compare? If they are different, why are they different?

## 2 Rotation with Constant Angular Acceleration

$\theta$  = angular displacement

$\omega$  = final angular velocity

$\omega_o$  = initial angular velocity

$\alpha$  = constant angular acceleration

$t$  = time

Here, we reintroduce the famous four equations for constant angular acceleration:

$$\omega = \omega_o + \alpha t \quad (1)$$

$$\theta = \frac{1}{2} (\omega + \omega_o) t = \bar{\omega} t \quad (2)$$

$$\theta = \omega t + \frac{1}{2} \alpha t^2 \quad (3)$$

$$\omega^2 = \omega_o^2 + 2\alpha\theta \quad (4)$$

**Ex. 14** A circular saw blade 0.200 m in diameter starts from rest. In 5.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

### 3 Relating Linear and Angular Kinematics

In this section, we introduce the relationship between *linear* and *angular* kinematic variables (for rigid bodies). Once we do this, we can use these relationships to determine the rotational dynamical quantities such as *rotational* kinetic energy ( $K_{\text{rot}}$ ).

Starting with the relationship  $s = r\theta$ , we can calculate the *time rate of change* of both sides of this equation:

$$\left| \frac{ds}{dt} \right| = r \left| \frac{d\theta}{dt} \right|$$

or

$$v = r\omega \quad (\text{relation between linear and angular speed})$$

Likewise, if we look at the *time rate of change* of both sides of this equation, we find the following relation:

$$\frac{dv}{dt} = r \frac{d\omega}{dt} \quad \text{or} \quad a_{\text{tan}} = r \alpha$$

There is a second acceleration, which we've seen before, namely  $a_{\text{rad}}$ .

$$a_{\text{rad}} = \frac{v^2}{r} = r\omega^2$$

**Ex. 24** Find the required angular speed (in rev/min) of an ultracentrifuge for the radial acceleration of a point 2.50 cm from the axis to equal 400,000g (That is, 400,000 times the acceleration due to gravity).

### 4 Energy in Rotational Motion

The kinetic energy of a point mass inside of a *rigid* body rotating about a *fixed* axis with angular velocity  $\omega$  is:

$$\frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

where  $v_i$  is the tangential velocity of the point mass.

The *total* kinetic energy of a rigid body rotating about a fixed axis is:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \dots = \sum_{i=1}^N \frac{1}{2}m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum_{i=1}^N m_i r_i^2 = \frac{1}{2} \omega^2 I$$

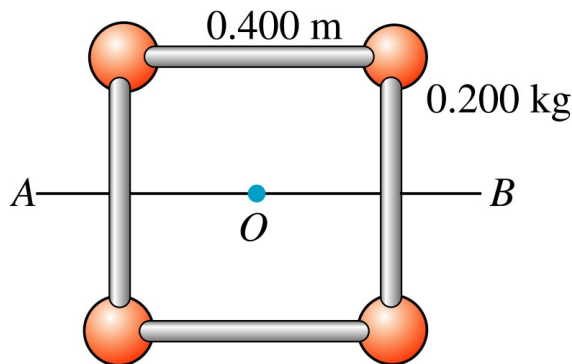
where we define the *moment of inertia* ( $I$ ):

$$I = \sum_{i=1}^N m_i r_i^2 \quad (\text{definition of the moment of inertia})$$

$$K = \frac{1}{2} I \omega^2 \quad (\text{rotational kinetic energy of a rigid body})$$

Let's look at the moments of inertia for various objects (Table 9.2).

**Ex. 34** Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by light rods (Fig. 9.27). Find the moment of inertia of the system about an axis a) through the center of the square, perpendicular to its plane (an axis through point  $O$  in the figure); b) bisecting two opposite sides of the square (an axis along the line  $AB$  in the figure); c) that passes through the centers of the upper left and lower right spheres and through point  $O$ .



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Figure 1: Fig. 9.27 from our textbook shows 4 point-masses forming a rigid body held together by 4 massless rods.

**Ex. 48** Suppose the solid cylinder in the apparatus described in Example 9.9 (Section 9.4) is replaced by a thin-walled, hollow cylinder with the same mass  $M$  and radius  $R$ . The cylinder is attached to the axle by spokes of a negligible moment of inertia. (a) Find the speed of the hanging mass  $m$  just as it strikes the floor. (b) Use energy concepts to explain why the answer to part (a) is different from the speed found in Example 9.9.

## 4.1 Gravitational Potential Energy for an Extended Body

We can calculate the gravitational potential energy for an *extended* body by summing up the potential energies for all the masses ( $m_i$ ) inside the body.

$$U_{\text{grav}} = m_1gy_1 + m_2gy_2 + m_3gy_3 + \cdots = g(m_1y_1 + m_2y_2 + m_3y_3 + \cdots)$$

However, we know from our *center-of-mass* equations:

$$m_1y_1 + m_2y_2 + m_3y_3 + \cdots = (m_1 + m_2 + m_3 + \cdots)y_{\text{cm}} = M y_{\text{cm}}$$

Thus,

$$U_{\text{grav}} = Mgy_{\text{cm}} \quad (\text{potential energy for an extended body}) \quad (5)$$

## 5 Parallel-Axis Theorem

If you know the moment of inertia of an object about an axis passing through its center-of-mass ( $I_{\text{cm}}$ ), then it's possible to *easily* calculate its new center-of-mass about another axis (that is parallel to the *original* axis). If the axis is translated a distance  $d$ , then the new moment of inertia can be written as:

$$I_{\text{new}} = I_{\text{cm}} + M d^2 \quad (\text{parallel-axis theorem}) \quad (6)$$

In many cases, calculating the moment of inertia requires doing an integral  $I = \int r^2 dm$ . Calculating the moment of inertia for an object rotating around a *new* fixed axis (parallel to the original axis) would normally require doing another integral  $I_{\text{new}} = \int r^2 dm$ . The parallel axis theorem states, that if you know the moment of inertia about the center-of-mass ( $I_{\text{cm}}$ ), then the new moment of inertia

for an axis parallel to the original axis can be easily determined without doing any integration.

Let's look at an example using the  $I_{cm}$ 's shown in Table 9.2.

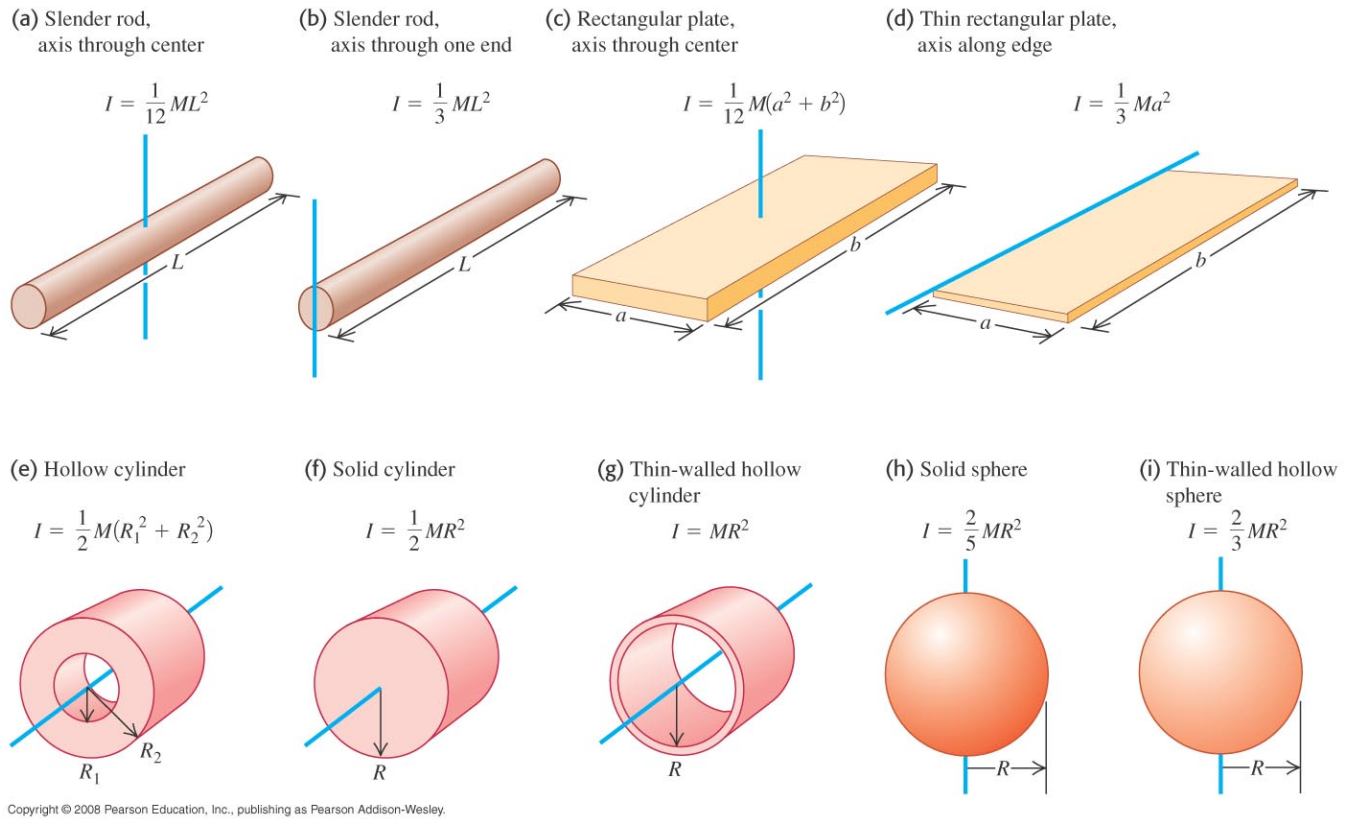


Figure 2: Moments of Inertia of Various Bodies

**Ex. 57** A thin, rectangular sheet of metal has mass  $M$  and sides of length  $a$  and  $b$ . Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

**Prob. 74** A sphere consists of a solid wooden ball of uniform density  $800 \text{ kg/m}^3$  and radius  $0.20 \text{ m}$  and is covered with a thin coating of lead foil with area density  $20 \text{ kg/m}^2$ . Calculate the moment of inertia of this sphere about an axis passing through its center.

## Homework – Chapter 9

Exercises: 2, 4, 10, 14, 24, 29, 34, 47, 48, 52, 57

Problems: 74, 79, 98