Chapter 7
Potential Energy and Energy Conservation

We saw in the previous chapter the relationship between *work* and *kinetic energy*. We also saw that the relationship was the same whether the net external force was constant or varying as a function of distance $F(x)$.

$$W_{\text{tot}} = \Delta K = K_{\text{f}} - K_{\text{i}} = \frac{1}{2}mv_{\text{f}}^2 - \frac{1}{2}mv_{\text{i}}^2$$

1 Gravitational Potential Energy

Let's calculate the work done by gravity as it acts on a system moving vertically in a gravitational field. When calculating the *work* done by a gravitational field, we must make sure that the $+y$ direction is pointing upward, away from the center of the earth.

$$W_{\text{grav}} = \vec{F} \cdot \vec{s} = mg\,y_1 - mg\,y_2$$

where y_1 and y_2 are the initial and final positions respectively.

Now we define the quantity mgy as the gravitational potential energy U_{grav}.

$$U_{\text{grav}} = mgy \qquad \text{(gravitational potential energy)}$$

$$W_{\text{grav}} = -(U_2 - U_1) = -\Delta U_{\text{grav}}$$

This equation is true in general for all forces where a potential energy function exists. The minus sign is *absolutely essential*.

N.B. If a potential energy function exists, we don't have to calculate the integral $W = \int \vec{F} \cdot d\vec{s}$.

N.B. The location of $U = 0$ is arbitrary. When using $W = -\Delta U$ in the work-energy theorem, the important feature is the *change* in potential (ΔU), not the absolute potential energy (U_1 or U_2).

Ex. 1 In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day.

1.1 Conservation of Mechanical Energy

Let's assume that the force due to gravity is the "only force" acting on a system (e.g., projectile motion). What can we learn by applying the work-energy theorem to such a system?

$$W = \Delta K \qquad \Rightarrow \qquad -\Delta U_{\rm grav} = \Delta K \qquad \Rightarrow \qquad -(U_2 - U_1) = K_2 - K_1$$

Rearranging terms we have:

$$K_1 + U_1 \; = \; K_2 + U_2 \qquad \text{(conservation of mechanical energy)} \tag{1}$$

or

$$\frac{1}{2}mv_1^2 + mgy_1 \; = \; \frac{1}{2}mv_2^2 + mgy_2 \quad \text{(if only gravity does the work)} \tag{2}$$

where the subscripts 1 and 2 represent the initial and final positions of the system respectively. We define the mechanical energy of the system as $E = K + U$. Again, **if gravity is the only force doing work on the system**, then the mechanical energy of the system is a *constant* of the motion (i.e., the mechanical energy is conserved).

$$E_1 = E_2 \quad \Rightarrow$$

$$\frac{1}{2}mv_1^2 + mgy_1 \; = \; \frac{1}{2}mv_2^2 + mgy_2 \quad \text{(Conservation of Mechanical Energy)}$$

1.2 Effect of Other Forces

What can we say if other forces besides gravity are performing work "on" the system? The work performed by "other" forces can be calculated using the work-energy theorem.

$$W_{\rm total} = \Delta K \qquad \Rightarrow \qquad W_{\rm other} + W_{\rm grav} = \Delta K \qquad W_{\rm other} - \Delta U = \Delta K$$

The work done "by" other forces becomes:

$$W_{\text{other}} = \Delta K + \Delta U = E_2 - E_1 \neq 0 \tag{3}$$

Example: Suppose a projectile is launched over level ground with an initial velocity of 40 m/s and it returns to the earth with a final velocity of 30 m/s. What is the work done by *air drag*?

1.3 Gravitational Potential Energy for Motion Along a Curved Path

Let's investigate what happens if we no longer restrict our motion to being purely in the y direction. Let's imagine an object moves in the x-y plane as shown in Fig 1 below. What is the work done "on" the system by gravity (i.e., $W_{\text{grav}} = ?$) ?

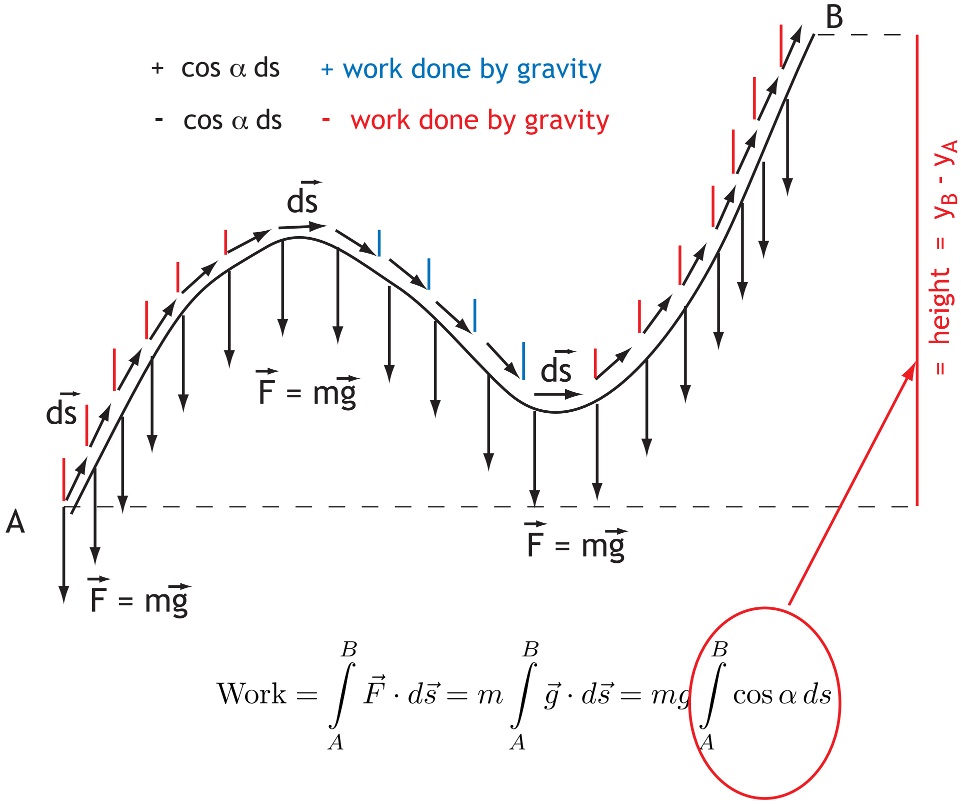

Figure 1: The vertical bars (red and blue) show the incremental length of $\cos\alpha\, ds$ for each step between point A and point B. The sum of the negative (red) bars and the positive (blue) bars results in a *net* length of red bars whose total length is the difference in height between points A and B.

In short, the total work done by gravity is still

$$W_{\text{grav}} = -\Delta U \;=\; -(mgy_B - mgy_A) \;=\; -mg(y_B - y_A) \;=\; -mg\,(\text{change in height})$$

N.B. The work done by gravity, even when an object has a complicated trajectory, only depends on the "change in height."

Ex. 11 You are testing a new amusement park roller coaster with an empty car with mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m/s and at the top of the loop (point B) has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

2 Elastic Potential Energy

Obviously, there are other forces besides gravity that can perform work "on" a system. Let's turn our attention to a force we saw in the previous chapter, namely the elastic force due to a spring. We saw that the work done "by" a spring is:

$$W_{\text{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \qquad \text{(work done "by" a spring)}$$

where 1 and 2 are the initial and final positions respectively. Notice the similarity between this equation to $W_{\text{grav}} = mgy_1 - mgy_2$. Again, the work done is the difference between two quantities (i.e., the potential energies). If we define

$$U_{\text{elastic}} = \frac{1}{2}kx^2$$

then we can write the work done "by" an elastic force as:

$$W_{\text{el}} = -\Delta U^{\text{elastic}} = -(U_2^{\text{el}} - U_1^{\text{el}}) \qquad (4)$$

Writing out each of the terms explicitly, we have:

$$W_{\text{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \qquad \text{(work done "by" a spring)}$$

Using the work-energy theorem, we have

$$W = \Delta K \quad \Rightarrow \quad W_{\text{el}} = K_2 - K_1 \quad \Rightarrow \quad -\Delta U_{\text{el}} = K_2 - K_1 \quad \Rightarrow$$

$$-(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad \Rightarrow \quad \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

Again, we see that we have conservation of mechanical energy, $E_1 = E_2$, where $E = K + U$, and the *elastic* force is the only external force doing work.

2.1 Situations with Both Gravitational and Elastic Potential Energy

Using the work-energy theorem, along with the potential energy functions we've defined thus far, make it trivial to combine the work done by multiple forces. Starting with the work-energy theorem, we have:

$$W = \Delta K \qquad \Rightarrow \qquad W_{\text{grav}} + W_{\text{el}} = \Delta K \tag{5}$$
$$-\Delta U_{\text{grav}} - \Delta U_{\text{el}} = \Delta K$$

Writing out each of the Δ's and combining terms, we have:

$$K_1 + U_1^{\text{grav}} + U_1^{\text{el}} = K_2 + U_2^{\text{grav}} + U_2^{\text{el}}$$

which becomes

$$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}kx_2^2 \tag{6}$$

N.B. We still have conservation of mechanical energy after combining the *elastic* force with the *gravitational* force, $E_1 = E_2$. The mechanical energy is again a "constant of the motion" (i.e., $E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}kx^2$ is the same throughout the motion).

Ex. 19 A spring of negligible mass has force constant $k = 1600$ N/m. a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? b) You place the spring vertically with one end on the floor. You then drop a 1.20-kg book onto it from a height of 0.80 m above the top of the spring. Find the maximum distance the spring will be compressed.

3 Conservative and Nonconservative Forces

In this section we're going to look at the left-hand side of the work-energy theorem and separate the forces doing the work into two different classes, *conservative* and *non-conservative*. First of all, how do we know if a force is a *conservative* force? The work done by a conservative force *always* has the following properties:

1. It can always be expressed as the difference between the initial and final values of a *potential energy* function.

2. It is reversible (i.e., mechanical energy related to that force is conserved).
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

What about *non-conservative* forces? The only way to calculate the work done "by" non-conservative forces is to return to our original definition of work, namely $W_{\text{nc}} = \int \vec{F} \cdot d\vec{s}$.

If all the forces are conservative, then:

$$E = K + U \qquad \text{and} \qquad E_1 = E_2$$

If one or more of the forces are non-conservative, then:

$$E = K + U \qquad \text{and} \qquad W_{\text{nc}} = E_2 - E_1$$

Example: A 1.00-kg box is released from rest on a frictionless inclined plane 2.00 meters above the ground. Once released, it slides down the incline and encounters a horizontal surface where the coefficient of kinetic friction is μ_k for a distance of two meters. After passing over the "friction pad," it proceeds up a frictionless incline until it reaches a maximum height of 1.50 meters. Find the coefficient of kinetic friction.

Example: A 1.00-kg box is released from rest on a frictionless inclined plane 2.00 meters above the ground. Once released, it slides down the incline and encounters a horizontal surface where the coefficient of kinetic friction is $\mu_k = 0.25$ for a distance of two meters. After passing over the "friction pad," it encounters a spring with a spring constant of 1600 N/m and compresses it. What is the maximum compression of the spring?

Ex. 28 In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2 \hat{\imath}$, where $\alpha = 12\ \mathrm{N/m^2}$. a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point (0.10 m, 0) to the point (0.10 m, 0.40 m)? b) Along the straight-line path from the point (0.10 m, 0) to the point (0.30 m, 0)? c) Along the straight-line path from the point (0.30 m, 0), to the point (0.10 m, 0)? d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential energy function for it? Let $U = 0$ when $x = 0$.

3.1 The Law of Conservation of Energy

The law of *conservation of energy* even encompasses the work done by non-conservative forces. For example, if we have non-conservative forces contributing the total work, we can write:

$$W = \Delta K \qquad \Rightarrow \qquad W_{\text{cons}} + W_{\text{nc}} = \Delta K \qquad \Rightarrow \qquad -\Delta U + W_{\text{nc}} = \Delta K$$

and from this, determine the total work done by non-conservative forces:

$$W_{\text{nc}} = \Delta K + \Delta U \qquad \Rightarrow \qquad W_{\text{nc}} = (K_2 + U_2) - (K_1 + U_1) = E_2 - E_1$$

In thermodynamics, it's possible to relate the W_{nc} to the change in internal energy of the working gas $(-\Delta U_{\text{int}})$. Thus

$$W_{\text{nc}} = -\Delta U_{\text{int}} = E_2 - E_1$$

or

$$\Delta U_{\text{int}} + \Delta K + \Delta U_{\text{cons}} = 0 \qquad \text{(law of conservation of energy)}$$

4 Force and Potential Energy

In physics, you will encounter situations where the potential energy as a function of position is known and you will want to find the corresponding force. We know from our definition of work that:

$$\Delta W = F_x\,\Delta x = -\Delta U \qquad \Rightarrow \qquad F = -\frac{\Delta U}{\Delta x}$$

As we take the limit $\Delta x \to 0$, we obtain the following equation:

$$F_x(x) = -\frac{dU}{dx} \qquad \text{(force from potential energy, one dimension)} \tag{7}$$

Ex. 32 The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

4.1 Force and Potential Energy in Three Dimensions

We can continue this definition of force into 2 and 3 dimensions if we know the potential energy function $U(x, y, z)$. The x, y, and z components of force are:

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

The force vector can be written as:

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\right) \qquad \text{(force from the potential energy)}$$

In short-hand notation, this equation can be written as:

$$\vec{F} = -\vec{\nabla}U \qquad \text{where} \qquad \vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

where $\vec{\nabla}$ is called the *del* operator, and $\vec{\nabla}U$ is the *gradient* of U.

Example: Suppose we have a three-dimensional harmonic oscillator whose potential energy function is $U(x, y, z) = \frac{1}{2}k(x^2 + y^2 + z^2)$. Find the force vector corresponding to this potential energy function.

5 Energy Diagrams

You can also plot the potential energy function $U(x)$ and locate *stable* and *unstable* equilibrium. These figures are called energy diagrams.

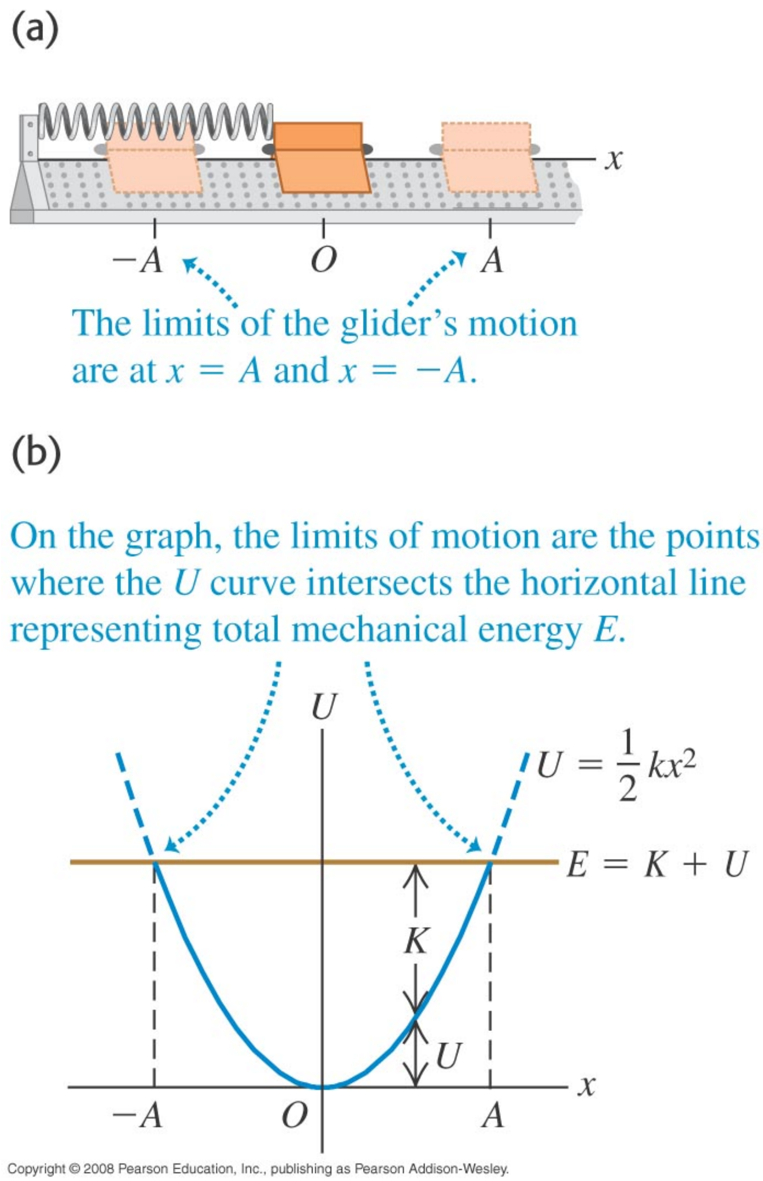

Figure 2: Potential energy function for a simple harmonic oscillator.

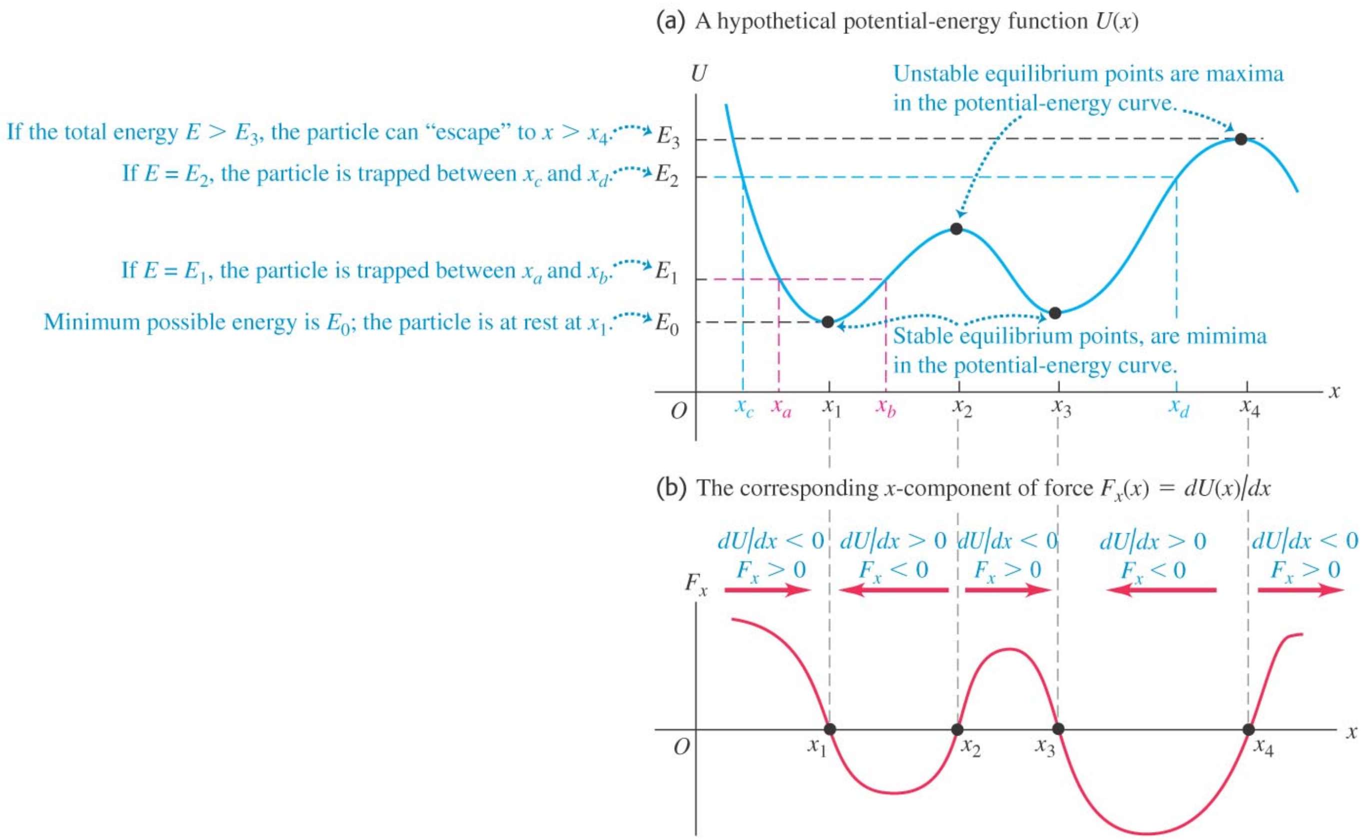

Figure 3: The relationship between the potential-energy function and the force for a hypothetical potential-energy function $U(x)$.

Prob. 58 A wooden rod of negligible mass and length 80.0 cm is pivoted about a horizontal axis through its center. A white rat with mass 0.500 kg clings to one end of the stick, and a mouse with mass 0.200 kg clings to the other end. The system is released from rest with the rod horizontal. If the animals can manage to hold on, what are their speeds as the rod swings through a vertical position?

Prob. 77 A particle of mass m is acted on by a conservative force and moves along a path given by $x = x_o \cos \omega_o t$ and $y = y_o \sin \omega_o t$, where x_o, y_o and ω_o are constants. a) Find the components of the force that acts on the particle. b) Find the potential energy of the particle as a function of x and y. Take $U = 0$ when $x = 0$ and $y = 0$. c) Find the total energy of the particle when: (i) $x = x_o$, $y = 0$ (ii) $x = 0$, $y = y_o$.

Homework – Chapter 7

Exercises: 1, 5, 11, 14, 16, 19, 25, 28, 32, 34, 38
Problems: 47, 53, 58, 74, 77