

Origins of the law of gravitation

1. Up to this point, we have discussed forces in terms of *push* and *pull* contact forces—one body pushing or pulling on another.
2. Gravity is a force that exerts its influence over a distance, sometimes called *action at a distance*.
3. Ptolemy (A.D. 2nd century) introduced the notion of epicycles to explain the orbits of the planets. It was unnecessarily complicated.
4. Nicolaus Copernicus (1473-1543) proposed a heliocentric scheme to describe the orbits of the planets.
5. Johannes Kepler (1571-1630) proposed three empirical laws describing the motions of the planets.
6. Isaac Newton (1642-1727) derived Kepler’s laws from his laws of mechanics and his proposed law of gravitation.
7. **Ockham’s Razor** is the principle proposed by William of Ockham in the fourteenth century: “Pluralitas non est ponenda sine neccesitate”, which translates as “entities should not be multiplied unnecessarily”.

Newton had his own version: “**Do not needlessly multiply causes.**”

Newton’s Law of Universal Gravitation

Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

$$F = G \frac{m_1 m_2}{r^2} \tag{1}$$

where G is called the *gravitational constant*. $G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$.

The Vector Force

$$\vec{\mathbf{F}}_{12} = -G \frac{m_1 m_2}{r^2} \hat{\mathbf{r}}_{12} \quad \text{the force on } m_1 \text{ due to } m_2.$$

$$\vec{\mathbf{F}}_{21} = -G \frac{m_1 m_2}{r_{21}^2} \hat{\mathbf{r}}_{21} \quad \text{the force on } m_2 \text{ due to } m_1.$$

where

$$\hat{\mathbf{r}}_{12} = \frac{\vec{r}_{12}}{r_{12}} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \quad \text{and}$$
$$\hat{\mathbf{r}}_{21} = \frac{\vec{r}_{21}}{r_{21}} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

Note that:

$$\vec{\mathbf{r}}_{21} = -\vec{\mathbf{r}}_{12} \quad \text{and} \quad \vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$$

Example: What is the force between a 2.00 kg mass and 5.00 kg mass separated by 20.0 cm center-to-center? (Assume both masses are spherical and homogeneous)

The Rule of Superposition

What is the force on m_1 if there are more than *two masses* present, for example, N masses?

It is the vector sum of all the *pair-wise* forces acting on m_1 .

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{14} + \cdots = \sum_{n=2}^N \vec{\mathbf{F}}_{1n}$$

Multiple choice 2. The magnitude of the force of gravity between two identical objects is given by F_o . If the mass of each object is doubled and the distance between them is also doubled, then the new force of gravity between the objects will be

- (A) $4F_o$ (B) $2F_o$ (C) F_o (D) $F_o/2$

Question 3. If the force of gravity acts on all bodies in proportion to their masses, why does a heavy body not fall correspondingly faster than a light body?

Exercise 2. How far from the Earth must a space probe be along a line toward the Sun so that the Sun's gravitational pull balances the Earth's?

Exercise 9. If g is to be determined by dropping an object through a distance of (exactly) 10 m, how accurately must the time be measured to obtain a result good to 0.1%? Calculate a percent error and an absolute error, in milliseconds.

The Gravitational Constant G

The first laboratory measurement of G was performed by Henry Cavendish in 1798. He used a torsion pendulum and measured a value of $6.75 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ with an uncertainty of $\pm 0.010 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$.

This same technique is used today and the accepted value of G is

$$G = 6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$$

The Mass of the Earth

Once the gravitational constant G is determined, the mass of the Earth can be calculated.

$$F = mg_o \quad \text{and} \quad F = \frac{GmM_E}{R_E^2}$$
$$M_E = \frac{g_o R_E^2}{G} = \frac{(9.83\text{m/s}^2) (6.37 \times 10^6\text{m})^2}{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2} = 5.98 \times 10^{24} \text{ kg}$$

Gravitation Near the Earth's Surface

Assumptions:

1. The Earth is spherical.
2. The density depends only on the radial distance from its center.

The magnitude of the gravitational force acting on a particle of mass m is:

$$F = G \frac{M_E m}{r^2} \quad \text{and} \quad F = mg_o \quad \text{where} \quad g_o = \frac{G M_e}{r^2}$$

How does g_o vary with altitude? Table 14-1

How does g_o vary with latitude? Figure 14-6

The Real Earth

1. The Earth's crust is not uniform and has local density fluctuations.
2. The Earth is not a sphere. It is an *ellipsoid*.
3. The Earth is rotating.

What is the effect of the rotating earth? At the equator,

$$g_o - g = \omega^2 R_E = 0.034 \text{ m/s}^2$$

where $g_o = GM_E/r^2 = GM_E/R_E^2$ on the equator.

There's something about a sphere

The Two Shell Theorem

- Shell Theorem #1 *A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.*
- Shell Theorem #2 *A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside it.*

Some key steps in the derivation

1. the density of a thin spherical shell $\rho = M/V = M/4\pi R^2 t$
2. the force on m due to small mass (dm_A) on the sphere

$$dF = G \frac{m dm_A}{x^2} (-\cos \alpha)$$

3. the force on m due to an annular ring of mass on the sphere

$$dF = G \frac{m dM}{x^2} (-\cos \alpha)$$

where dM is the mass in the annular ring

$$dM = \rho dV = \rho [2\pi(R \sin \theta) (R d\theta) t]$$

where t is the thickness of the *thin* spherical shell.

4. the relationship between θ and α

$$\cos \alpha = \frac{r - R \cos \theta}{x} \quad (\text{outside the spherical shell})$$

$$\cos \alpha = \frac{R \cos \theta - r}{x} \quad (\text{inside the spherical shell})$$

5. the law of cosines

$$x^2 = r^2 + R^2 - 2rR \cos \theta$$

6. making x the only independent variable

$$\sin \theta d\theta = \frac{x}{rR} dx$$

$$dF = -\frac{\pi G t \rho m R}{r^2} \left(\frac{r^2 - R^2}{x^2} + 1 \right) dx$$

7. **Outside the spherical shell**

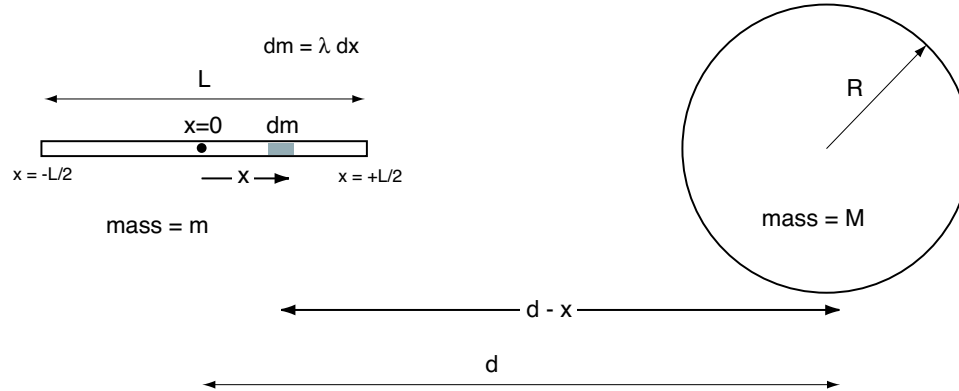
$$F = \int dF = -\frac{\pi G t \rho m R}{r^2} \int_{r-R}^{r+R} \left(\frac{r^2 - R^2}{x^2} + 1 \right) dx = -\frac{\pi G t \rho m R}{r^2} (4R)$$

8. **Inside the spherical shell**

$$F = \int dF = -\frac{\pi G t \rho m R}{r^2} \int_{R-r}^{R+r} \left(\frac{-r^2 + R^2}{x^2} + 1 \right) dx = -\frac{\pi G t \rho m R}{r^2} (0)$$

Example

Suppose we have a *uniform* solid sphere of mass M and a *thin* cylindrical bar of mass m separated by a distance d between their respective centers of mass.



$$dF = \frac{GM \lambda dx}{(d-x)^2} \quad F = GM\lambda \int_{-L/2}^{+L/2} \frac{dx}{(d-x)^2} = \frac{GMm}{d^2} \frac{1}{\left(1 - \frac{L^2}{4d^2}\right)}$$

where dF is the force on dm , and $dF = GM dm/(d-x)^2$, $dm = \lambda dx$, and $m = \lambda L$.

The total force on the rod (F) of length L is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass.

Problem What is the total force on the rod if it is rotated 90° about its center-of-mass?

$$dF_x = dF \cos \alpha = \frac{GM\lambda dy}{(d^2 + y^2)} \cos \alpha = \frac{GM\lambda d dy}{(d^2 + y^2)^{3/2}}$$

$$F = GM\lambda d \int_{-L/2}^{L/2} \frac{dy}{(d^2 + y^2)^{3/2}} = \frac{GMm}{d^2} \frac{1}{\sqrt{1 + \frac{L^2}{4d^2}}}$$

Again, the total force on the rod of length L is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass.

The *sphere* appears to be the only “simple” object where one can calculate the force by concentrating all its mass at the center-of-mass.

Another application of the shell theorem

Suppose a hole is drilled through the earth M_E passing through its center. What is the force on a mass m as a function of r , its distance from the center of the earth?

$$F(r) = -\frac{GM_E m}{R_E^3} r = -kr \quad (\text{Hooke's Law})$$

The mass m undergoes *simple harmonic motion* similar to a mass on a spring. We'll study this later in chapter 17.

How does g vary with r , the distance from the center of the earth?

Graviational potential energy

1. Potential energy for a mass at a single point.

$$F = -\frac{dU}{dr} \quad dU = -F dr \quad \int_{\infty}^r dU = - \int_{\infty}^r F dr = - \int_{\infty}^r \frac{-GMm}{r^2} dr$$

$$U(r) - U(\infty) = U(r) = -\frac{GMm}{r}$$

2. The change in potential energy between two points.

$$\Delta U_{\text{grav}} = U_{\text{final}} - U_{\text{initial}} = -\frac{GMm}{r_f} - \left(-\frac{GMm}{r_i}\right)$$

Work-Energy Theorem

$$W = \Delta KE$$

However, since gravity is a conservative force (we have a PE function, U_{grav})

$$W_{\text{grav}} = -\Delta U_{\text{grav}} = GMm \left(\frac{1}{r_f} - \frac{1}{r_i} \right) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Applications of gravitational potential energy

1. Calculate the escape velocity

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

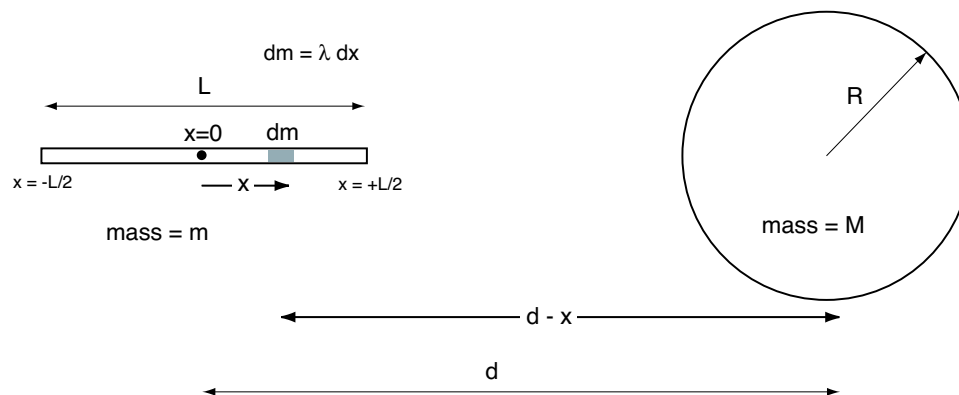
2. Calculate the potential energy of *many particle* systems. For 3 particles,

$$U = - \left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}} \right)$$

Since U is negative, this is called the *binding energy*. It takes this much energy to remove these particles infinitely far from each other such that they arrive at ∞ with zero *kinetic energy*.

3. Gravitational Gradient Alignment

Let's go back to our problem where we have a thin, cylindrical bar of mass m and a uniform sphere of mass M separated by a distance d . What is the potential energy of this two-mass system?



$$dU = - \frac{GM \lambda dx}{(d-x)} \quad U_{\text{horizontal}} = -GM\lambda \int_{-L/2}^{+L/2} \frac{dx}{(d-x)} = - \frac{GMm}{L} \ln \left(\frac{1 + \frac{L}{2d}}{1 - \frac{L}{2d}} \right)$$

where dU is the potential energy between dm and M , and $dU = -GM dm/(d-x)$, and $dm = \lambda dx$, and $m = \lambda L$.

The total potential energy (U) between the rod of length L and the sphere of mass M is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass. That is, $U \neq -GMm/d$

Problem What is the total potential energy if the rod is rotated 90° about its center-of-mass?

$$dU = - \frac{GM dm}{\sqrt{d^2 + y^2}} = - \frac{GM \lambda dy}{(d^2 + y^2)^{1/2}}$$

$$U_{\text{vertical}} = -GM\lambda \int_{-L/2}^{L/2} \frac{dy}{(d^2 + y^2)^{1/2}} = -\frac{GMm}{L} \left(2 \sinh^{-1} \left(\frac{L}{2d} \right) \right)$$

Examining the two results (U_{vertical} and $U_{\text{horizontal}}$), we can see that

$$U_{\text{horizontal}} < U_{\text{vertical}}$$

for all values of $L/2d$ where $d > L/2$. The rod prefers to be in the *horizontal* orientation compared to the *vertical* orientation.

The motions of planets and satellites

We can understand the motion of bodies in the solar system using Newton's laws of motion and the law of universal gravitation. First, we will make two assumptions to simplify the analysis.

1. We will consider the gravitational force only between the orbiting body (e.g., the earth) and the central body (the Sun)
2. We will assume that the central body is so much more massive than the orbiting body that we can ignore its motion under their mutual interaction.

Kepler's laws of motion are empirical, but can be derived using Newton's laws of motion and the universal law of gravitation.

1. All planets move in elliptical orbits having the Sun at one focus.

$$\begin{array}{ll} \textit{perigee}\text{-point of orbit that is closest to the Sun} & R_p = a(1 - e) \\ \textit{apogee}\text{-point of orbit that is furthest from the Sun} & R_a = a(1 + e) \end{array}$$

a is the semi-major axis of the ellipse, and e is the eccentricity.

2. A line joining any planet to the Sun sweeps out equal areas in equal times.

$$\frac{dA}{dt} = \frac{L}{2m}$$

3. The square of the period of any planet about the Sun is proportional to the cube of the planet's mean distance from the Sun.

$$T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3$$

Table 14-3 *Kepler's Law of Periods for the Solar System*

Energy considerations in planetary and satellite motion

The total mechanical energy of a planet m moving around the Sun M is:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

However, from Newton's 2nd Law, and assuming a *circular* orbit,

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \text{and} \quad mv^2 = \frac{GMm}{r}$$

So, the total mechanical energy becomes:

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

What if the orbit is elliptical instead of circular? Then the total mechanical energy of the orbit is:

$$E = K + U = -\frac{GMm}{2a}$$

where a is the semi-major axis of the ellipse.

Total energy considerations

1. $E < 0$ The two masses (M and m) are *bound* and m pursues an elliptical orbit about M .
2. $E = 0$ $K = -U$ and m has just enough kinetic energy to achieve the *escape velocity* and pursues a parabolic orbit around M coming to rest at $r = \infty$ in an infinite amount of time.
3. $E > 0$ $K > U$ and m is no longer *bound* to M and pursues a hyperbolic orbit around M never to return.

Is Newton's law of universal gravitation complete?

The answer to this question is, "No." A more complete theory of gravitation is required to explain the following phenomena:

1. The precession of the perihelion of Mercury (43 arc-seconds per century).
2. The bending of light near a massive object does not follow a straight line.
3. Radar echoes are delayed as they pass close to the Sun.

Einstein's theory of *general relativity* is required to explain these phenomena. In this theory, the larger mass M sets up a field (*curved space*) and the second mass m (or massless object, e.g., a photon) pursues its motion as it interacts with the field established by the mass M .

The presence of matter causes the geometry of space to "curve." Likewise, the geometry directs the object how it should move through that region of space.

Exercise 37 Assume that a geosynchronous communications satellite is in orbit at the longitude of Chicago. You are in Chicago and want to pick up its signals. In what direction should you point the axis of your parabolic antenna? The latitude of Chicago is 47.5° N.

Problem 8 The fastest possible rate of rotation of a planet is that for which the gravitational force on material at the equator barely provides the centripetal force needed for the rotation. (Why?)

(a) Show then that the corresponding shortest period of rotation is given by

$$T = \sqrt{\frac{3\pi}{G\rho}}$$

(b) Evaluate the rotation period assuming a density of 3.0 g/cm^3 .

Problem 24 Two point-like particles, each of mass m , are originally at rest separated by a distance d . Show that the time for them to come together under the influence of gravity is

$$t_{\text{meet}} = \frac{\pi}{4} \sqrt{\frac{d^3}{Gm}}$$

Problem 27 In the year 1610, Galileo made a telescope, turned it on Jupiter, and discovered four prominent moons. Their mean orbit radii a and periods T are as follows

Name	$a(10^8 \text{ m})$	T (days)
Io	4.22	1.77
Europa	6.71	3.55
Ganymede	10.7	7.16
Callisto	18.8	16.7

- Plot $\log a$ (y axis) against $\log T$ (x axis) and show that you get a straight line.
- Measure its slope and compare it with the value that you expect from Kepler's law of periods.
- Find the mass of Jupiter from the intercept of this line with the y axis.

Note: You may also use log-log graph paper

