

# Chapter 7

## Potential Energy and Energy Conservation

We saw in the previous chapter the relationship between *work* and *kinetic energy*. We also saw that the relationship was the same whether the net external force was constant or varying as a function of distance  $F(x)$ .

$$W_{\text{tot}} = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

### 1 Gravitational Potential Energy

Let's calculate the work done by gravity as it acts on a system moving vertically in a gravitational field. When calculating the *work* done by a gravitational field, we must make sure that the  $+y$  direction is pointing upward, away from the center of the earth.

$$W_{\text{grav}} = \vec{F} \cdot \vec{s} = mg y_1 - mg y_2$$

where  $y_1$  and  $y_2$  are the initial and final positions respectively.

Now we define the quantity  $mg y$  as the gravitational potential energy  $U_{\text{grav}}$ .

$$U_{\text{grav}} = mg y \quad (\text{gravitational potential energy})$$

$$W_{\text{grav}} = -(U_2 - U_1) = -\Delta U_{\text{grav}}$$

This equation is true in general for all forces where a potential energy function exists. The minus sign is *absolutely essential*.

**N.B.** If a potential energy function exists, we don't have to calculate the integral  $W = \int \vec{F} \cdot d\vec{s}$ .

**N.B.** The location of  $U = 0$  is arbitrary. When using  $W = -\Delta U$  in the work-energy theorem, the important feature is the *change* in potential ( $\Delta U$ ), not the absolute potential energy ( $U_1$  or  $U_2$ ).

**Ex. 1** In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day.

## 1.1 Conservation of Mechanical Energy

Let's assume that the force due to gravity is the "only force" acting on a system (e.g., projectile motion). What can we learn by applying the work-energy theorem to such a system?

$$W = \Delta K \quad \Rightarrow \quad -\Delta U_{\text{grav}} = \Delta K \quad \Rightarrow \quad -(U_2 - U_1) = K_2 - K_1$$

Rearranging terms we have:

$$K_1 + U_1 = K_2 + U_2 \quad (\text{conservation of mechanical energy}) \quad (1)$$

or

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \quad (\text{if only gravity does the work}) \quad (2)$$

where the subscripts 1 and 2 represent the initial and final positions of the system respectively. We define the mechanical energy of the system as  $E = K + U$ . Again, **if gravity is the only force doing work on the system**, then the mechanical energy of the system is a *constant* of the motion (i.e., the mechanical energy is conserved).

$$E_1 = E_2 \quad \Rightarrow$$

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \quad (\text{Conservation of Mechanical Energy})$$

## 1.2 Effect of Other Forces

What can we say if other forces besides gravity are performing work "on" the system? The work performed by "other" forces can be calculated using the work-energy theorem.

$$W_{\text{total}} = \Delta K \quad \Rightarrow \quad W_{\text{other}} + W_{\text{grav}} = \Delta K \quad W_{\text{other}} - \Delta U = \Delta K$$



**N.B.** The work done by gravity, even when an object has a complicated trajectory, only depends on the “change in height.”

**Ex. 11** You are testing a new amusement park roller coaster with an empty car with mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m/s and at the top of the loop (point B) has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

## 2 Elastic Potential Energy

Obviously, there are other forces besides gravity that can perform work “on” a system. Let’s turn our attention to a force we saw in the previous chapter, namely the elastic force due to a spring. We saw that the work done “by” a spring is:

$$W_{\text{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \quad (\text{work done “by” a spring})$$

where 1 and 2 are the initial and final positions respectively. Notice the similarity between this equation to  $W_{\text{grav}} = mgy_1 - mgy_2$ . Again, the work done is the difference between two quantities (i.e., the potential energies). If we define

$$U_{\text{elastic}} = \frac{1}{2}kx^2$$

then we can write the work done “by” an elastic force as:

$$W_{\text{el}} = -\Delta U^{\text{elastic}} = -(U_2^{\text{el}} - U_1^{\text{el}}) \quad (4)$$

Writing out each of the terms explicitly, we have:

$$W_{\text{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \quad (\text{work done “by” a spring})$$

Using the work-energy theorem, we have

$$\begin{aligned} W = \Delta K \quad \Rightarrow \quad W_{\text{el}} = K_2 - K_1 \quad \Rightarrow \quad -\Delta U_{\text{el}} = K_2 - K_1 \quad \Rightarrow \\ -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad \Rightarrow \quad \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2 \end{aligned}$$

Again, we see that we have conservation of mechanical energy,  $E_1 = E_2$ , where  $E = K + U$ , and the *elastic* force is the only external force doing work.

## 2.1 Situations with Both Gravitational and Elastic Potential Energy

Using the work-energy theorem, along with the potential energy functions we've defined thus far, make it trivial to combine the work done by multiple forces. Starting with the work-energy theorem, we have:

$$\begin{aligned} W = \Delta K &\quad \Rightarrow \quad W_{\text{grav}} + W_{\text{el}} = \Delta K \\ -\Delta U_{\text{grav}} - \Delta U_{\text{el}} &= \Delta K \end{aligned} \tag{5}$$

Writing out each of the  $\Delta$ 's and combining terms, we have:

$$K_1 + U_1^{\text{grav}} + U_1^{\text{el}} = K_2 + U_2^{\text{grav}} + U_2^{\text{el}}$$

which becomes

$$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}kx_2^2 \tag{6}$$

**N.B.** We still have conservation of mechanical energy after combining the *elastic* force with the *gravitational* force,  $E_1 = E_2$ . The mechanical energy is again a “constant of the motion” (i.e.,  $E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}kx^2$  is the same throughout the motion).

**Ex. 21** A spring of negligible mass has force constant  $k = 1600$  N/m. a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? b) You place the spring vertically with one end on the floor. You then drop a 1.20-kg book onto it from a height of 0.80 m above the top of the spring. Find the maximum distance the spring will be compressed.

## 3 Conservative and Nonconservative Forces

In this section we're going to look at the left-hand side of the work-energy theorem and separate the forces doing the work into two different classes, *conservative* and *non-conservative*. First of all, how do we know if a force is a *conservative* force? The work done by a conservative force *always* has the following properties:

1. It can always be expressed as the difference between the initial and final values of a *potential energy* function.

2. It is reversible (i.e., mechanical energy related to that force is conserved).
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

What about *non-conservative* forces? The only way to calculate the work done “by” non-conservative forces is to return to our original definition of work, namely  $W_{\text{nc}} = \int \vec{F} \cdot d\vec{s}$ .

**If all the forces are conservative, then:**

$$E = K + U \quad \text{and} \quad E_1 = E_2$$

**If one or more of the forces are non-conservative, then:**

$$E = K + U \quad \text{and} \quad W_{\text{nc}} = E_2 - E_1$$

**Example:** A 1.00-kg box is released from rest on a frictionless inclined plane 2.00 meters above the ground. Once released, it slides down the incline and encounters a horizontal surface where the coefficient of kinetic friction is  $\mu_k$  for a distance of two meters. After passing over the “friction pad,” it proceeds up a frictionless incline until it reaches a maximum height of 1.50 meters. Find the coefficient of kinetic friction.

**Example:** A 1.00-kg box is released from rest on a frictionless inclined plane 2.00 meters above the ground. Once released, it slides down the incline and encounters a horizontal surface where the coefficient of kinetic friction is  $\mu_k = 0.25$  for a distance of two meters. After passing over the “friction pad,” it encounters a spring with a spring constant of 1600 N/m and compresses it. What is the maximum compression of the spring?

**Ex. 28** In an experiment, one of the forces exerted on a proton is  $\vec{F} = -\alpha x^2 \hat{i}$ , where  $\alpha = 12 \text{ N/m}^2$ . a) How much work does  $\vec{F}$  do when the proton moves along the straight-line path from the point (0.10 m, 0) to the point (0.10 m, 0.40 m)? b) Along the straight-line path from the point (0.10 m, 0) to the point (0.30 m, 0)? c) Along the straight-line path from the point (0.30 m, 0), to the point (0.10 m, 0)? d) Is the force  $\vec{F}$  conservative? Explain. If  $\vec{F}$  is conservative, what is the potential energy function for it? Let  $U = 0$  when  $x = 0$ .

### 3.1 The Law of Conservation of Energy

The law of *conservation of energy* even encompasses the work done by non-conservative forces. For example, if we have non-conservative forces contributing the total work, we can write:

$$W = \Delta K \quad \Rightarrow \quad W_{\text{cons}} + W_{\text{nc}} = \Delta K \quad \Rightarrow \quad -\Delta U + W_{\text{nc}} = \Delta K$$

and from this, determine the total work done by non-conservative forces:

$$W_{\text{nc}} = \Delta K + \Delta U \quad \Rightarrow \quad W_{\text{nc}} = (K_2 + U_2) - (K_1 + U_1) = E_2 - E_1$$

In thermodynamics, it's possible to relate the  $W_{\text{nc}}$  to the change in internal energy of the working gas ( $-\Delta U_{\text{int}}$ ). Thus

$$W_{\text{nc}} = -\Delta U_{\text{int}} = E_2 - E_1$$

or

$$\Delta U_{\text{int}} + \Delta K + \Delta U_{\text{cons}} = 0 \quad (\text{law of conservation of energy})$$

## 4 Force and Potential Energy

In physics, you will encounter situations where the potential energy as a function of position is known and you will want to find the corresponding force. We know from our definition of work that:

$$\Delta W = F_x \Delta x = -\Delta U \quad \Rightarrow \quad F = -\frac{\Delta U}{\Delta x}$$

As we take the limit  $\Delta x \rightarrow 0$ , we obtain the following equation:

$$F_x(x) = -\frac{dU}{dx} \quad (\text{force from potential energy, one dimension}) \quad (7)$$

**Ex. 32** The potential energy of a pair of hydrogen atoms separated by a large distance  $x$  is given by  $U(x) = -C_6/x^6$ , where  $C_6$  is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

**Example Gravity in One Dimension.** Two point masses,  $m_1$  and  $m_2$  lie on the  $x$ -axis with  $m_1$  held in place at the origin and  $m_2$  at position  $x$  and free to move. The gravitational potential energy of these masses is found to be  $U(x) = -Gm_1m_2/x$ , where  $G$  is a constant (called the **gravitational constant**). You'll learn more about gravitation in Chapter 12. Find the  $x$ -component of the force acting on  $m_2$  due to  $m_1$ . Is this force attractive or repulsive? How do you know?

## 4.1 Force and Potential Energy in Three Dimensions

We can continue this definition of force into 2 and 3 dimensions if we know the potential energy function  $U(x, y, z)$ . The  $x$ ,  $y$ , and  $z$  components of force are:

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

The force vector can be written as:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = -\left(\hat{i} \frac{\partial U}{\partial x} + \hat{j} \frac{\partial U}{\partial y} + \hat{k} \frac{\partial U}{\partial z}\right) \quad (\text{force from the potential energy})$$

In short-hand notation, this equation can be written as:

$$\vec{F} = -\vec{\nabla}U \quad \text{where} \quad \vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

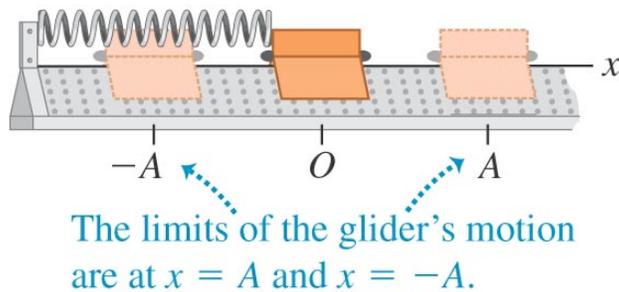
where  $\vec{\nabla}$  is called the *del* operator, and  $\vec{\nabla}U$  is the *gradient* of  $U$ .

**Example:** Suppose we have a three-dimensional harmonic oscillator whose potential energy function is  $U(x, y, z) = \frac{1}{2}k(x^2 + y^2 + z^2)$ . Find the force vector corresponding to this potential energy function.

## 5 Energy Diagrams

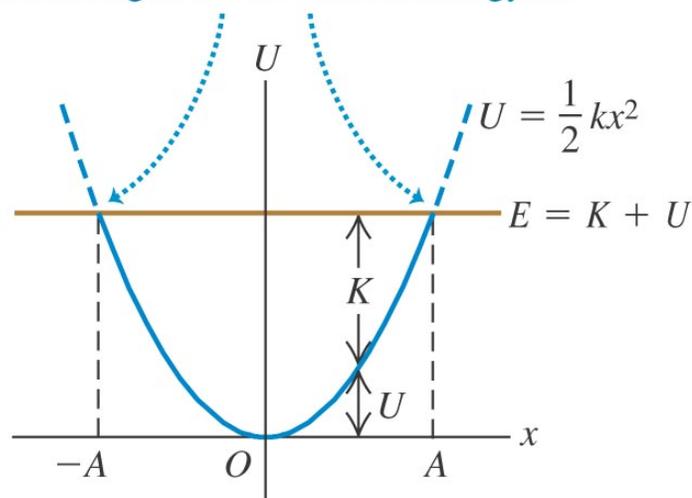
You can also plot the potential energy function  $U(x)$  and locate *stable* and *unstable* equilibrium. These figures are called energy diagrams.

(a)



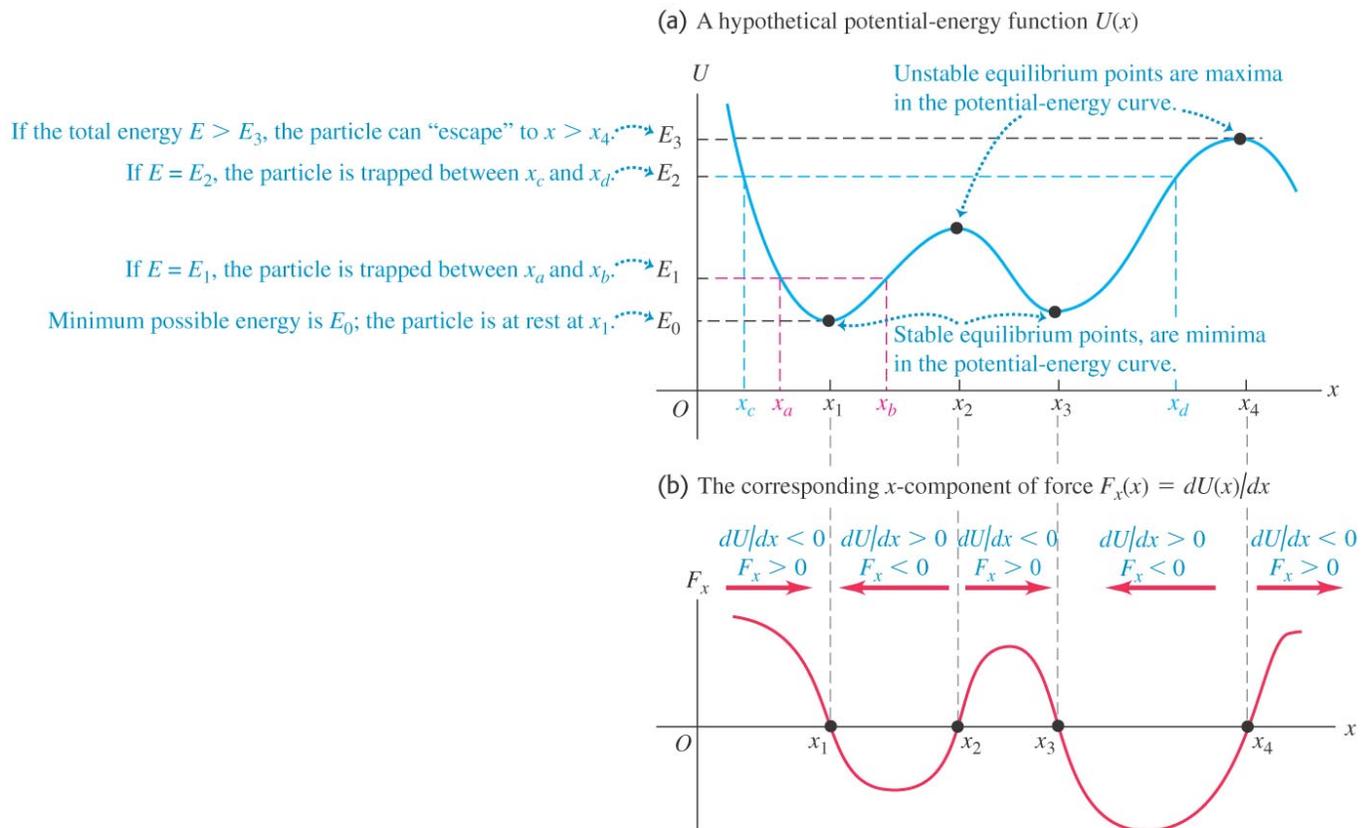
(b)

On the graph, the limits of motion are the points where the  $U$  curve intersects the horizontal line representing total mechanical energy  $E$ .



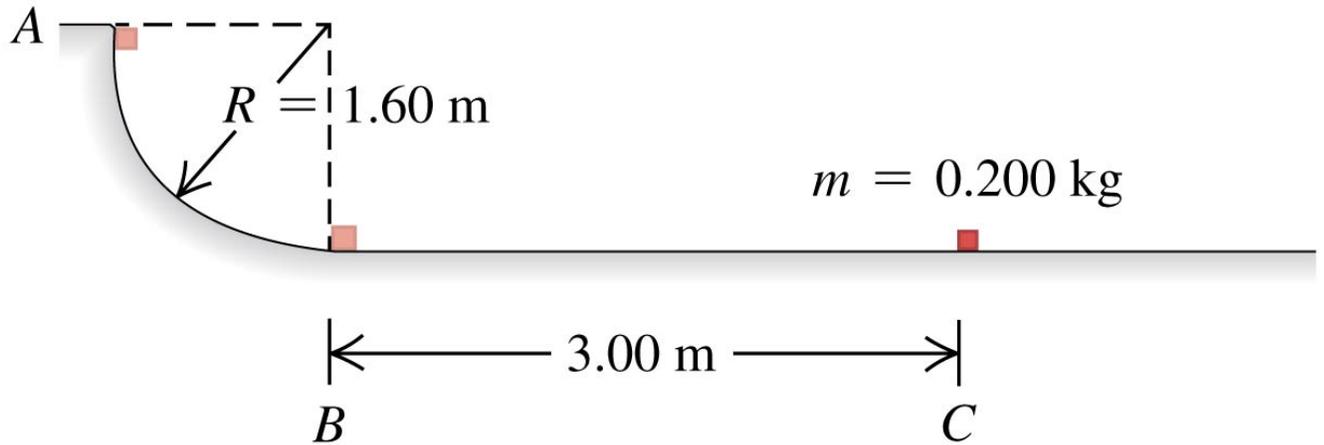
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Figure 2: Potential energy function for a simple harmonic oscillator.



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Figure 3: The relationship between the potential-energy function and the force for a hypothetical potential-energy function  $U(x)$ .



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Figure 4: This is the figure for problem 57 from chapter 7 of University Physics 14<sup>th</sup> edition.

**Prob. 57** In a truck-loading station at a post office, a small 0.200-kg package is released from rest at point  $A$  on a track that is one-quarter of a circle with radius 1.60 m (**Fig. P7.57**). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point  $B$  with a speed of 4.80 m/s. From point  $B$ , it slides on a level surface a distance of 3.00 m to point  $C$ , where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from  $A$  to  $B$ .

**Prob. 70** A small block with mass 0.0400 kg slides in a vertical circle of radius  $R = 0.500 \text{ m}$  on the inside of a circular track. During one of the revolutions of the block, when the block is at the bottom of its path, point  $A$ , the magnitude of the normal force exerted on the block by the track has magnitude 3.95 N. In this same revolution, when the block reaches the top of its path, point  $B$ , the magnitude of the normal force exerted on the block has magnitude 0.680 N. How much work was done by friction during the motion of the block from point  $A$  to point  $B$ ?